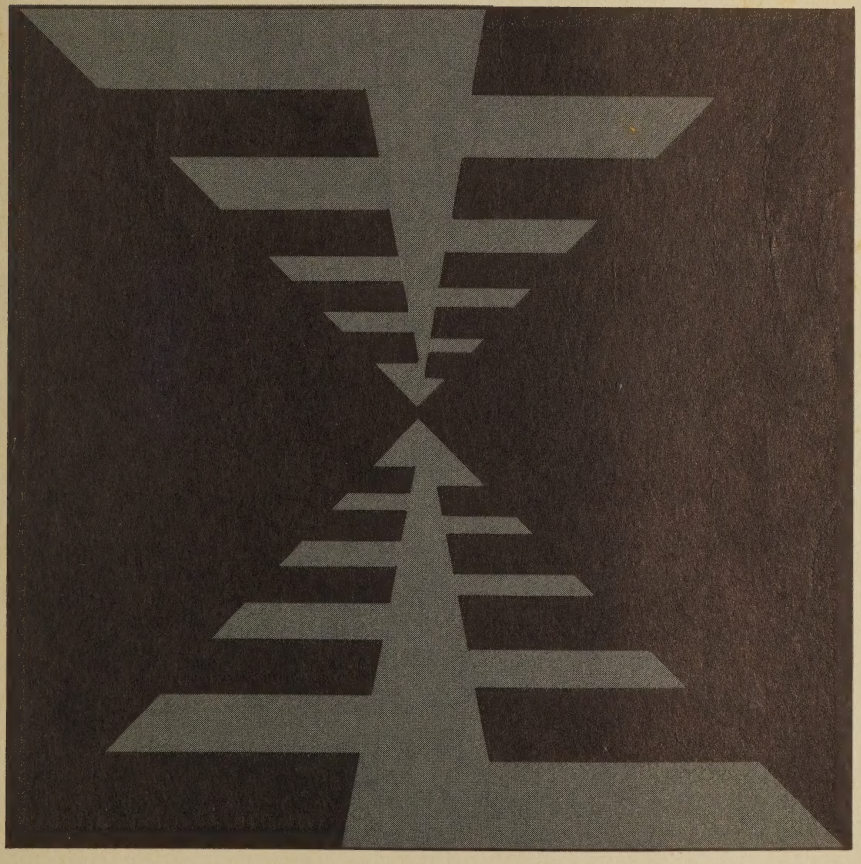
FOR INSTRUCTORS ONLY

JAMES B. BEYER

Solutions to Problems in

ANTENNA ENGINEERING

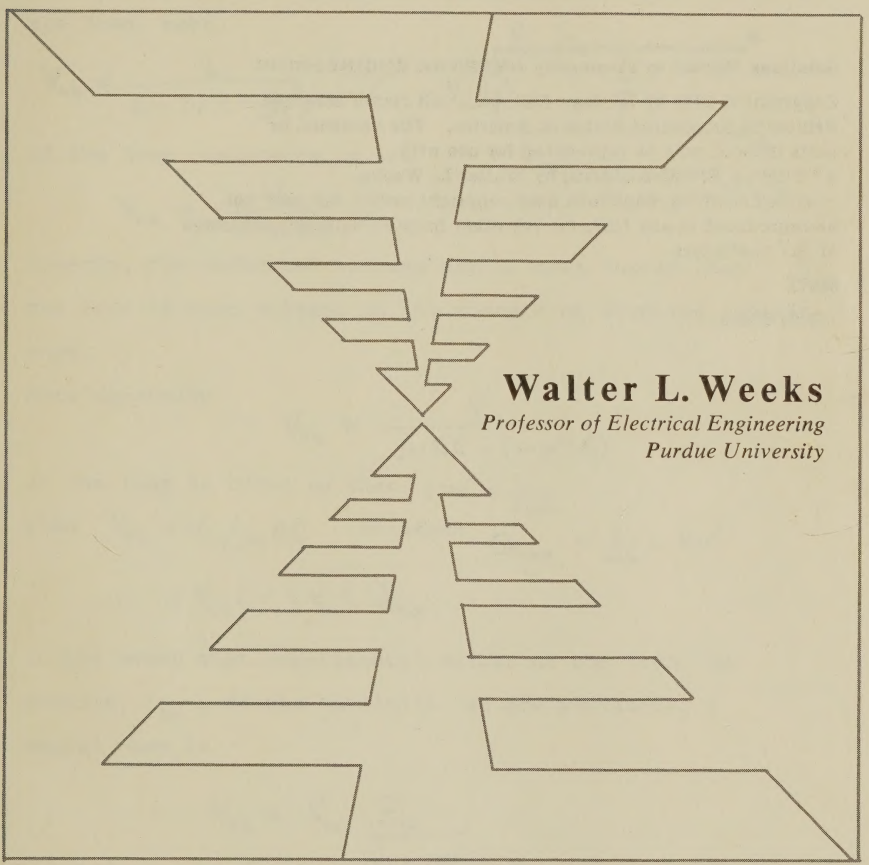
Walter L. Weeks



McGraw-Hill Book Company

Solutions to Problems in

ANTENNA ENGINEERING



Walter L. Weeks
Professor of Electrical Engineering
Purdue University

McGraw-Hill Book Company
New York St. Louis San Francisco
Toronto London Sydney

Solutions Manual to accompany ANTENNA ENGINEERING

Copyright © 1968 by McGraw-Hill, Inc. All rights reserved. Printed in the United States of America. The contents, or parts thereof, may be reproduced for use with ANTENNA ENGINEERING, by Walter L. Weeks, provided such reproductions bear copyright notice, but may not be reproduced in any form for any other purpose without permission of the publishers.

68971

1234WH70698

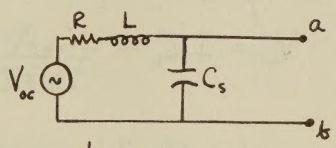
Prob.1.1. If there is a shunting capacitance, the currents in the loop are not zero even with a perfect voltmeter. Maxwell's equation, integrated, is

$$j\omega\mu \iint H \cdot dS = V_{ab} + \int \frac{i}{\sigma} dl = -j\omega\mu [H_x + H_i] \Delta S$$

where H_x is the field to be measured and H_i is the field produced by the currents in the loop.

Looking into the terminals of the loop, note

$$V_{ab} = \frac{V_{oc}}{R + j\omega\left(L - \frac{1}{\omega^2 C}\right)} \cdot \frac{1}{j\omega C}$$



$$V_{oc} = j\omega\mu H_x \Delta S$$

If the loop resistance is low

$$V_{ab} \doteq -\frac{V_{oc}}{1 - \omega^2 LC}$$

Clearly, the indicated voltage may be much larger than the open circuit voltage in the absence of shunting capacitance.

More generally

$$V_{ab} = \frac{V_{oc}}{j\omega RC + (1 - \omega^2 LC)}$$

If the loop is tuned so that $j\omega C = \frac{1}{j\omega L}$

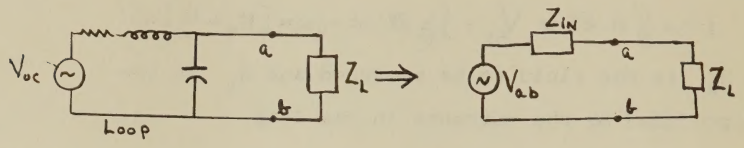
then $V_{ab} = V_{oc}/j\omega RC$ or since $Q_{loop} = \frac{R}{\omega L} = R\omega C$

$$|V_{ab}| = |V_{oc}| Q_{loop}$$

In the event that experimental values of the input impedance, Z_{in}, to the terminals ab are available, a useful form is

$$V_{ab} = V_{oc} \frac{Z_{in}}{R + j\omega L}$$

Prob.1.1,part 2) . If a measuring device with impedance Z_{load} is connected to terminals ab, we may calculate the effects from an equivalent circuit, as follows:



The voltage, V_M , measured across Z_L is

$$V_M = I Z_L = \frac{V_{ab} Z_L}{Z_{IN} + Z_L} = \frac{V_{ab}}{1 + \frac{Z_{IN}}{Z_L}}$$

or

$$V_M = \frac{V_{oc} Z_{IN}}{R + j\omega L} \cdot \frac{1}{\left[1 + \frac{Z_{IN}}{Z_L}\right]}$$

$$= \frac{j\omega\mu H_x \Delta S}{R + j\omega L} \frac{1}{\left[\frac{1}{Z_{IN}} + \frac{1}{Z_L}\right]}$$

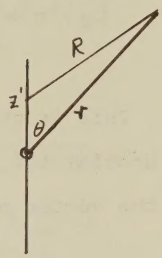
This is the indicated voltage, given in terms of the i input impedance looking into the connector leading to the loop, the input impedance of the amplifier, and the field strength.

Problem 1.2 The reader should parallel the developments presented in Section 1.5, and so obtain the results stated.

Problem 1.3

The vector potential produced by a uniform line of current along the z-axis is (eq. 18)

$$A_z = \frac{1}{4\pi}\int_{-h}^{h} \frac{e^{-j\beta R}}{R} I(z')\,dz'$$



$$R = r - z'\cos\theta$$

$$A_z = \frac{e^{-j\beta r}}{4\pi r} I_o \int_{-h}^{h} e^{-j\beta z'\cos\theta}\,dz'$$

$$= \frac{e^{-j\beta r}}{4\pi r} I_o \left[\frac{e^{j\beta h\cos\theta} - e^{-j\beta h\cos\theta}}{j\beta\cos\theta}\right]$$

$$= \frac{e^{-j\beta r}}{4\pi r} I_o h \frac{\sin(\beta h\cos\theta)}{\beta h\cos\theta}$$

In the distant field region

$$E_\theta = j\omega\mu A_z \sin\theta = j\frac{\omega\mu}{2\pi}\frac{e^{-j\beta r}}{r} I_o h \sin\theta \frac{\sin(\beta h\cos\theta)}{\beta h\cos\theta}$$

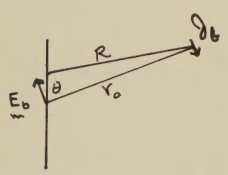
Problem 1.4

Let the line source be called source a. Introduce a test source $\underset{\sim}{J}_b = \underset{\sim}{\hat{\theta}}\,\delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$

By the reciprocity theorem

$$E_\theta(x_o,y_o,z_o) = \int_{-h}^{h} \underset{\sim}{E}_b \cdot \underset{\sim}{\hat{z}}\, I_a(z)\,dz$$

The field produced by the point dipole with reference to its own coordinate system (primed) is



$$\underset{\sim}{E}_b = \underset{\sim}{\hat{\theta}}'\, j\frac{\omega\mu}{4\pi R} e^{-j\beta R} \sin\theta'$$

Note $\theta' = \frac{\pi}{2}$ for all positions

$$R = r_o - z\cos\theta$$

Prob 1.4 (cont.)

$$\text{Also } \underset{\sim}{E}_b \cdot \hat{\underset{\sim}{z}} = E_b \sin\theta$$

so

$$E_\theta(x_0, y_0, z_0) = \frac{j\omega\mu}{4\pi r_0} e^{-j\beta r_0} \sin\theta \, I_a \int_{-h}^{h} e^{-j\beta z \cos\theta} \, dz$$

This is the same definite integral that was evaluated in problem 1.3. The result is then the same as that obtained by the vector potential method in problem 1.3

Problem 1.5 To find the radial component of the electric field produced by an arbitrary current distribution, $\underset{\sim}{J}_a$ set up a radially oriented unit point test dipole, $\underset{\sim}{J}_b$, in the distant field region, and apply the reciprocity theorem

$$\int \underset{\sim}{E}_a \cdot \underset{\sim}{J}_b \, dv = \int \underset{\sim}{E}_b \cdot \underset{\sim}{J}_a \, dv$$

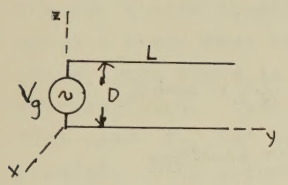
or since $\underset{\sim}{J}_b$ is a unit point source, radially directed

$$E_{a,r} = \int \underset{\sim}{E}_b \cdot \underset{\sim}{J}_a \, dV$$

But in the direction toward the source (the origin), the distant field of the test dipole, E_b, is zero since a point dipole has no r-component in the distant field. Therefore $E_{a,r} = 0$. QED

no field along the axis of the point dipole

Prob. 2.1



First find the current in the structure.

By eq. 7

$$I(y) = -\frac{V_g}{Z_o}\,\frac{\sinh\gamma(y-L)}{\cosh\gamma L}$$

The current at the input is $I(0) = \frac{V_g}{Z_{in}} = \frac{V_g}{Z_o \coth\gamma L}$

so $$I(y) = \frac{-I(0)}{\tanh\gamma L}\,\frac{\sinh\gamma(y-L)}{\cosh\gamma L} = -\frac{I(0)\sinh\gamma(y-L)}{\sinh\gamma L}$$

Now we may calculate the radiation from the vertical wire in the direction $\theta = \frac{\pi}{2}$

$$A_z = \frac{I(0)\,D\,e^{-j\beta r_o}}{4\pi r_o}\,, \qquad E_\theta = \frac{j\omega\mu\, I(0)\,D\,e^{-j\beta r_o}}{4\pi r_o}$$

and the radiation from the transmission line pair:

$$A_y = \frac{1}{4\pi}\int_0^L \left(\frac{e^{-j\beta R_1}}{R_1} - \frac{e^{-j\beta R_2}}{R_2}\right) I(y)\,dy\,.$$

For points in the direction $\theta = 0$ (along the z axis)

$$R_1 = r_o - \frac{D}{2}\,, \quad R_2 = r_o + \frac{D}{2}$$

or

$$A_y = \frac{-e^{-j\beta r_o}}{4\pi r_o} I(0)\int_0^L \left(e^{j\frac{\beta D}{2}} - e^{-j\frac{\beta D}{2}}\right)\frac{\sinh\gamma(y-L)}{\sinh\gamma L}\,dy$$

$$= -\frac{I(0)\,e^{-j\beta r_o}}{4\pi r_o}\,\frac{2j\sin\frac{\beta D}{2}}{\sinh\gamma L}\,\frac{1}{\gamma}\cosh\gamma(y-L)\Big]_0^L$$

$$= -\frac{I(0)\,e^{-j\beta r_o}}{2\pi r_o}\,\frac{j\sin\frac{\beta D}{2}}{\sinh\gamma L}\,\frac{1}{\gamma}\left[1-\cosh\gamma L\right]$$

We are assuming that D is small so $\sin\frac{\beta D}{2} \doteq \frac{\beta D}{2}$ and

$$A_y = -\frac{I(0)}{4\pi r_o}\,e^{-j\beta r_o}\,\frac{j\beta D}{\sinh\gamma L}\,\frac{1}{\gamma}\left[1-\cosh\gamma L\right]$$

Prob 2.1 cont'.

For the direction under consideration

$$E_y = j\omega\mu A_y = \frac{\omega\mu I(0)}{4\pi r_0} e^{-j\beta r_0} \frac{\beta D}{\sinh \gamma L} \frac{1}{\gamma} [1 - \cosh \gamma L]$$

So, comparing the fields in the specified two directions

$$\frac{E_\theta]_{\theta=\frac{\pi}{2}}}{E_y]_{\theta=0}} = \frac{\gamma \sinh \gamma L}{\beta [1 - \cosh \gamma L]}$$

As a specific case, consider $L = \lambda/4$, $\gamma L = \beta L \left(j + \frac{\alpha}{\beta}\right)$
or $\gamma L = \frac{\pi}{2}(j + .01)$

$\sinh \gamma L = j \cosh .005\pi + 0$
$\cosh \gamma L = 0 + j \sinh .005\pi$ $\qquad \frac{\gamma}{\beta} \doteq j + .01$

Then

$$\frac{E_\theta]_{\theta=\frac{\pi}{2}}}{E_y]_{\theta=0}} = \frac{(j+.01)\, j \cosh .005\pi}{1 - j \sinh .005\pi} \doteq \frac{(j+.01)\, j}{1 - j.0157}$$

or

$$\left| \frac{E_\theta]_{\theta=\frac{\pi}{2}}}{E_y]_{\theta=0}} \right| \doteq 1$$

For shorter lengths $\sinh \gamma L \doteq \gamma L$, $1 - \cosh \gamma L \doteq \frac{(\gamma L)^2}{2}$

so

$$\frac{E_\theta]_{\theta=\frac{\pi}{2}}}{E_y]_{\theta=0}} \doteq \frac{\gamma}{\beta} \frac{\gamma L\, 2}{(\gamma L)^2} \doteq \frac{2}{\beta L}$$

However, for such short lengths, the current in the vertical wire would be non uniform, reducing its field by a factor of two in the limit. Thus for short lengths, L,

$$\frac{E_\theta]_{\theta=\frac{\pi}{2}}}{E_y]_{\theta=0}} \doteq \frac{1}{\beta L}$$

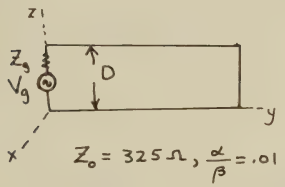
Prob. 2.1 cont'

The net radiation pattern is the superposition of the two fields considered. The transmission line pair gives a field that has a null in the $z = 0$ ($\theta = \pi/2$) plane; the field in that plane is then just that of the vertical wire. Call the field generated by the transmission line pair $E_{\theta'}$ (θ' is a polar angle measured from the y-axis). With reference to the angle θ'

$$E_{\theta'} \sim \sin\theta' \sin\tfrac{1}{2}(\beta D \sin\theta')$$

In the y-z plane the components E_θ and $E_{\theta'}$ are codirectional and add. In the x-z plane, these components are perpendicular to each other.

Problem 2.2



Neglecting radiation, the input impedance is

$$Z_{in} = Z_o \tanh \gamma L$$

$$V = V_1 e^{\gamma y} + V_2 e^{-\gamma y}$$

$$I = Y_o \left[V_2 e^{-\gamma y} - V_1 e^{\gamma y} \right]$$

At $y = L$, $V = 0$, so $V_1 = -V_2 e^{-2\gamma L}$

$$I(0) = Y_o [V_2 - V_1] = \frac{V_g}{Z_g + Z_o \tanh \gamma L}$$

$$V_2 \left[1 + e^{-2\gamma L}\right] = V_g Z_o \Big/ Z_g + Z_o \tanh \gamma L$$

$$V_1 = -\frac{V_g Z_o}{Z_g + Z_o \tanh \gamma L} \frac{e^{-2\gamma L}}{\left[1 + e^{-2\gamma L}\right]}$$

so

$$I(y) = \frac{V_g}{(Z_g + Z_o \tanh \gamma L)} \frac{1}{\left[1 + e^{-2\gamma L}\right]} \left[e^{-\gamma y} + e^{\gamma (y - 2L)} \right]$$

Prob. 2.2 cont'.

Considering the feeding vertical wire and the terminating short circuit as the primary radiators

$$I(0) = \frac{V_g}{Z_g + Z_0 \tanh \gamma L} \quad , \quad I(L) = \frac{V_g}{Z_g + Z_0 \tanh \gamma L} \cdot \frac{2 e^{-\gamma L}}{\left[1 + e^{-2\gamma L}\right]}$$

Case I

$$L = \frac{\lambda}{4} \quad , \quad \gamma L = \beta \frac{\lambda}{4} \left(\frac{\alpha}{\beta} + j\right) = \frac{\pi}{2}(.01 + j) \quad , \quad 2\gamma L = \pi(.01 + j)$$

$$e^{-\gamma L} = -j\, e^{-.01\frac{\pi}{2}} \quad , \quad e^{-2\gamma L} = -1\, e^{-.01\pi}$$

$$\tanh \gamma L = \frac{1 + e^{-.01\pi}}{1 - e^{-.01\pi}} \quad , \quad e^{-.01\pi} \doteq 1 - .01\pi \; ; \; e^{-.01\frac{\pi}{2}} \doteq 1 - .01\frac{\pi}{2}$$

$$\text{so } \tanh \gamma L \doteq \frac{2 - .01\pi}{.01\pi} \doteq 63$$

$$I(L) = I(0) \frac{2(-j)(1 - .01\frac{\pi}{2})}{\left[1 - (1 - .01\pi)\right]} = -j\, I(0) \left[\frac{200}{\pi} - 1\right] = -j\, I(0)\, 63 .$$

If $Z_g \ll Z_0$, $I(0) \doteq \frac{V_g}{325.63}$; if $Z_g = Z_0$, $I(0) = \frac{V_g}{325.64}$

The total vector potential is

$$A_z = \frac{I(0)\, D\, e^{-j\beta r}}{4\pi r} - \frac{I(L)\, D\, e^{-j\beta r_1}}{4\pi r_1}$$

$r_1 = r - L \sin\theta \sin\varphi$, so

$$A_z = \frac{I(0)\, D}{4\pi r} e^{-j\beta r} \left[1 + j63\, e^{j\frac{\pi}{2} \sin\theta \sin\varphi}\right]$$

$$E_\theta = j\omega\mu \sin\theta\, A_z .$$

This is essentially the pattern due to the current in the short circuit by itself.

An approximation for the input impedance is $Z_{in} = Z_0 \tanh \gamma L$ or $Z_{in} = 325.63$. The actual value would be somewhat less than this since the impedance of the short circuit is $R_r = 80\pi^2 (D/\lambda)^2$, . This value rather than zero should be transferred through a quarter wavelength to the input.

Prob 2.2 cont.[2]

The efficiency may be estimated as follows: The approximation for the input impedance is

$$\frac{Z_{in}}{Z_o} = \frac{R_r + Z_o \tanh \gamma L}{Z_o + R_r \tanh \gamma L} = \frac{R_r + Z_o 63}{Z_o + R_r 63}$$

$$\text{Power input} \doteq |I(0)|^2 R_{in} \doteq \frac{V_g}{Z_o 63} Z_o \frac{(R_r + 63 Z_o)}{(Z_o + 63 R_r)}$$

$$\text{The power radiated is} \doteq |I(L)|^2 R_r \doteq \frac{63 V_g^2}{Z_o 63} R_r$$

or

$$\text{Rad. Efficiency} = \frac{\frac{63 V_g^2 R_r}{Z_o}}{\frac{V_g^2}{63}\left(\frac{R_r + 63 Z_o}{Z_o + 63 R_r}\right)} = 63^2 \frac{R_r}{Z_o} \frac{(Z_o + 63 R_r)}{(R_r + 63 Z_o)}$$

$$\doteq 63^{\not 2} \frac{R_r}{Z_o} \frac{(Z_o + 63 R_r)}{\not 63 Z_o} \doteq 63 \frac{R_r}{Z_o}\left[1 + \frac{63 R_r}{Z_o}\right]$$

Case II

$$\frac{L}{\lambda} = \frac{1}{2}, \quad \gamma L = \pi(.01 + j), \quad 2\gamma L = 2\pi(.01 + j)$$

$$e^{-\gamma L} \doteq -1 + .01\pi \; ; \quad e^{-2\gamma L} = 1 - .02\pi$$

$$\tanh \gamma L = \frac{1 - e^{-2\gamma L}}{1 + e^{-2\gamma L}} = \frac{.02\pi}{2 - .02\pi} \doteq .01\pi$$

$$I(L) = I(0) \cdot 2 \cdot \frac{(-1 + .01\pi)}{2 - .02\pi} \doteq -I(0)$$

The vector potential is

$$A_z = \frac{I(0)\, D\, e^{-j\beta r}}{4\pi r}\left[1 + e^{j\pi \sin\theta \sin\varphi}\right]$$

$$E_\theta = j\omega\mu \sin\theta\, A_z$$

Note that in the $\theta = \frac{\pi}{2}$ plane, the bracket is

$$\left[1 + e^{j\pi \sin\theta \sin\varphi}\right] = 1 + e^{j\pi \sin\varphi} = e^{j\frac{\pi}{2}\sin\varphi}\left[2\cos\left(\frac{\pi}{2}\sin\varphi\right)\right]$$

In the absence of radiation the input impedance would be

$$Z_{in} = 325 \times .01\pi \doteq 10.2\ \Omega$$

The input impedance neglecting mutual effects would be approximately

$$Z_{in} \doteq 2\left(\frac{80\pi^2}{}\right)\left(D/\lambda\right)^2 + 10.2$$

With this approximation, the efficiency would be

$$\text{Rad. Eff.} = \frac{2(80\pi^2)(D/\lambda)^2}{2(80\pi^2)(D/\lambda)^2 + 10.2} \doteq \frac{1}{1 + \frac{10.2}{160\pi^2(\frac{D}{\lambda})^2}}$$

Case III

$L \ll \frac{\lambda}{4}$ so $\tanh \gamma L \doteq \gamma L$

$$I(L) \doteq I(0)\frac{2(1-\gamma L) + \frac{(\gamma L)^2}{2}}{1 + 1 - 2\gamma L + \frac{(2\gamma L)^2}{2}} = I(0)\frac{\left[1 - \gamma L + \frac{(\gamma L)^2}{4}\right]}{\left[1 - \gamma L + (\gamma L)^2\right]}$$

$$\doteq I(0)\left[1 - \frac{3}{4}\frac{(\gamma L)^2}{(1-\gamma L)}\right] \doteq I(0)\left[1 - \tfrac{3}{4}(\gamma L)^2\right]$$

Thus, the vector potential is

$$A_z = \frac{I(0)\,D}{4\pi r} e^{-j\beta r}\left[\frac{3}{4}(\gamma L)^2 e^{j\beta L \sin\theta \sin\varphi}\right]$$

$$E_\theta = j\omega\mu \sin\theta\, A_z$$

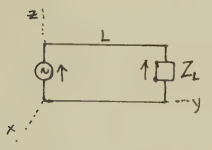
The input impedance is approximately $Z_{in} \doteq 325 \cdot \gamma L$

$$R_{in} = 325\alpha L = 3.25\beta L$$

$$\text{Rad. Efficiency} \doteq \frac{80\pi^2(\frac{D}{\lambda})^2 \cdot \frac{9}{16}(\gamma L)^4}{3.25\beta L + 80\pi^2(\frac{D}{\lambda})^2 \frac{9}{16}|\gamma L|^4}$$

If in the case that L is a half wavelength, the shorting wire end is bent around and placed near the input, the current in the feed wire and shorting wire are nearly equal in amplitude and phase, so they radiate the pattern of a single short dipole but with twice the current moment. (This neglects mutual effects; see Chap. 4 for better solution). Thus the pattern is simpler and more omnidirectional.

Problem 2.3



The object is to adjust Z_L so that the phase of the current in the feedline is the same as that in Z_L.

The current at any point on the line is

$$I(y) = I_2\left(e^{-\gamma y} + \rho_2 e^{\gamma(y-2L)}\right)$$

Where $\rho_2 = \dfrac{Y_L - Y_0}{Y_L + Y_0}$, $Y_L = \dfrac{1}{Z_L}$. The current through Z_L is $-I(L)$. To make the current in feed line and load equal in magnitude and phase, we want $\dfrac{I(0)}{I(L)} = -1$

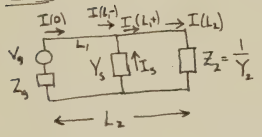
$$\frac{I(0)}{I(L)} = \frac{1+\rho_2 e^{-2\gamma L}}{e^{-\gamma L} + \rho_2 e^{-\gamma L}} = e^{\gamma L}\frac{(1+\rho_2 e^{-2\gamma L})}{(1+\rho_2)} = -1$$

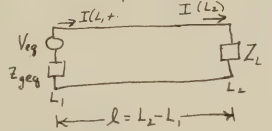
or $1 + \rho_2 e^{-2\gamma L} = -e^{-\gamma L} - \rho_2 e^{-\gamma L}$, $\rho_2 e^{-\gamma L}(e^{-\gamma L} + 1) = -(1+e^{-\gamma L})$

or $\rho_2 = -e^{+\gamma L}$ or $\dfrac{Y_L - Y_0}{Y_L + Y_0} = \dfrac{-(Z_L - Z_0)}{Z_L + Z_0} = -e^{+\gamma L}$

or $Z_L(1 - e^{\gamma L}) = Z_0(1 + e^{\gamma L})$ or $\dfrac{Z_L}{Z_0} = -\coth\dfrac{\gamma L}{2}$. Thus for a nearly lossless line, the load required to make the current in Z_L and in the feed line the same is $Z_L = jZ_0 \cot\dfrac{\beta L}{2}$, on a low loss line.

Part II Consider the addition of a third vertical radiator. This can be handled by breaking into parts:





Right section equivalent

From the equivalent, we have for the right most section

$$I(y) = I_3\left[e^{-\gamma y} + \rho_2 e^{\gamma(y-2\ell)}\right], \text{ so } \frac{I(L_1+)}{I(L_2)} = \frac{1+\rho_2 e^{-2\gamma\ell}}{e^{-\gamma\ell} + \rho_2 e^{-\gamma\ell}}$$

Define $\tau_2 = \dfrac{I(L_1+)}{I(L_2)}$. Then $\tau_2 e^{-\gamma\ell} + \tau_2\rho_2 e^{-\gamma\ell} = 1 + \rho_2 e^{-\gamma\ell}$,

From which

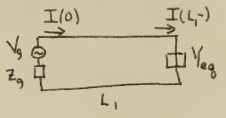
$$\rho_2 = e^{\gamma\ell}\frac{(1 - \tau_2 e^{-\gamma\ell})}{\tau_2 - e^{-\gamma\ell}}$$

Prob 2.3 cont.²

$$\frac{Y_2}{Y_0} = \frac{1+\rho_2}{1-\rho_2} \quad \text{so after a little algebra}$$

$$\frac{Y_2}{Y_0} = \frac{\sinh \gamma \ell}{r_2 - \cosh \gamma \ell}$$

For the left section, the mechanics of the analysis is similar



Define $r_1 = \frac{I(0)}{I(L_1^-)} = \frac{1+\rho_1 e^{-2\gamma L_1}}{e^{-\gamma L_1} + \rho_1 e^{\gamma L_1}}$

By analogy with the previous result

$$\frac{Y_{eq}}{Y_0} = \frac{\sinh \gamma L_1}{r_1 - \cosh \gamma L_1}$$

Now let us specify that the currents in the three vertical wires be the same in magnitude and phase:

$$I(0) = I_s = -I(L_2)$$

where also $I(L_1^-) = I(L_1^+) - I_s$ or $\frac{I(L_1^+)}{I(L_1^-)} = 1 + \frac{I_s}{I(L_1^-)} = 1 + r_1$

Note also $r_1 r_2 = \frac{I(0)}{I(L_1^-)} \cdot \frac{I(L_1^+)}{I(L_2)} = (-1)\frac{I(L_1^+)}{I(L_1^-)}$ or

$$r_1 r_2 = (-1)(1+r_1), \quad r_1(r_2+1) = -1 \quad \text{or}, \quad r_1 = \frac{-1}{r_2+1}$$

This result implies that the only restrictions on r_1 and r_2 are that they must satisfy this equation with realizable impedances.

Let us also insist that Y_s and Y_2 both be pure imaginary except for their (neglible) radiation resistance. Given that $\gamma \doteq j\beta$, we have

$$\frac{Y_2}{Y_0} = \frac{j \sin \beta L}{r_2 - \cos \beta L}, \quad Y_{eq} = Y_s + Y_0 \frac{[Y_2 + jY_0 \tan \beta \ell]}{[Y_0 + jY_2 \tan \beta \ell]} = \frac{Y_0 \sin \beta L_1}{r_1 - \cos \beta L_1}$$

These equations fix Y_2 and Y_s and they are imaginary if r_1 and r_2 are real.

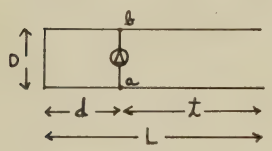
An interesting special case is $r_2 = 0, r_1 = -1$, so that

$$Y_2 = -jY_0 \tan \beta \ell, \quad Y_s = -jY_0 \tan \frac{\beta L_1}{2}.$$

Problem 2.4

This problem illustrates a method that has been used to match the impedance of small antennas to typical transmission line impedances. The main part of the solution should be done graphically, with the aid of the Smith Chart. The graphical procedure illustrates the general idea of the method and also points up some of the practical difficulties. The difficulty of reading the numbers on the chart is closely related to the "touchiness" of this type of structure.

As a preliminary, consider the transmission line alone, neglecting any radiation. The input admittance at the points ab is



$$Y = Y_o \coth \gamma d + Y_o \tanh \gamma t$$

By known identity formulas (for example those in Dwight's Tables #655.3 and #655.4)

$$\frac{Y_{in}}{Y_o} = \frac{\sinh 2\alpha d - j \sin 2\beta d}{\cosh 2\alpha d - \cos 2\beta d} + \frac{\sinh 2\alpha t + j \sin 2\beta t}{\cosh 2\alpha t + \cos 2\beta t}$$

Inspection shows that if $L = t+d = \lambda/4$, then the input susceptance will be zero if $\cosh 2\alpha t = \cosh 2\alpha d$. This is exactly satisfied if $d = t$; if α is small, it is practically true for any values of d and t since

$$\cosh 2\alpha t = 1 + \tfrac{1}{2}(2\alpha t)^2 \quad , \quad \cosh 2\alpha d = 1 + \tfrac{1}{2}(2\alpha d)^2$$

In the graphical solution, the Smith Chart is employed to transfer the admittances to ab. The analytic facts mentioned above become apparent since the short circuit is transferred a distance d and the open circuit is transferred a distance $t \doteq \lambda/4 - d$, to obtain zero susceptance.

As a better approximation, consider the effect of the small radiation resistance and the small inductance of the "short circuit". For a short length, D, the radiation resistance will be only of the order of an ohm or less and the reactance will be similarly very small. Thus, $G_r/Y_o \approx 600$ ohms, so on the chart the point is very nearly at the $\rho^I = +1$ value. Exactly what the large value is makes little difference in the graphical analysis.

The impedance transformation of the "short circuit" lies along the curve $\rho = |\rho_o|\, e^{-2\gamma d} = |\rho_o|\, e^{-2(.02)\beta d}\, e^{-j2\beta d}$. The impedance transformation of the open circuit lies

Prob.2.4 cont'd.

along the curve $\rho = -1\ e^{-2\beta t(.02-j)}$. Thus, the first step is to plot these gentle exponential spirals on the Smith Chart as a function of d/λ and t/λ ; viz.

$$\rho = |\rho_o| e^{-4\pi(.02)\frac{d}{\lambda}}\ e^{-j4\pi\frac{d}{\lambda}}$$

Having done this, the admittance values under the corresponding points d/λ and $t/\lambda = \frac{1}{4} - d/\lambda$ are read off and added. Note that the susceptance sums to zero quite accurately except at very small values of d. Also note that with very small values of d it is hard to read the admittances, particularly the conductances, accurately.

After plotting out the admittance values, and studying the curves on the Smith Chart, it appears by inspection that the two normalized conductances will add to the value 2 at $d \doteq .022\lambda$. The reactances don't quite cancel at that point but at $t = \frac{1}{4}\lambda - .0215\lambda$, the reactances do cancel. Thus, the $\frac{1}{4}\lambda$ line might need a little "trimming" to obtain a normalized admittance at the points ab of $G_{in}/Y_o = 2$, so that $Z_{in}/Z_o = \frac{1}{2}$ or $Z_{in} = 300$ ohms pure real as required.

Note the importance of the loss in the line to the desired solution.

. A plot of admittance as a function of frequency, obtained in the same way, shows the very narrow bandwidth of such structures.

Problem 2.5

With an antenna 10 meters in height limited to 30 meters in diameter, we can get a rough idea of the capacitance to ground with the parallel plate formula:

$C = \epsilon_o A/h = \frac{10^{-9}}{36\pi}\ \pi(15)^2/10 = .625 \times 10^{-9}$

The capacitive reactance at 100 KHz is then $X_c = -2520$ ohms. Thus the tuning coil would require an inductance such that $L = 4$ millihenries.

(At this point the student should ask himself if he knows how to design a coil such that it will have the required inductance, obtaining in the process a good estimate of the loss resistance, and thereby the coil Q. If an honest answer is that he does not know how to do it, he

Prob. 2.5 cont.
should inform himself, for example by consulting a handbook or reference book such as Terman's " Radio Engineering Handbook" or the ITT "Reference Data for Radio Engineers").

An approximation for the radiation resistance is

$R_r = 1600(h/\lambda)^2 = 1600(10/3000)^2 = 1.8 \times 10^{-2}$ ohms.

The big question is now " how does this compare with the coil losses?" To get an idea of the coil losses, suppose we candesign easily a coil having a Q of 100. Then $R_{coil} = \omega L/Q = 25.2$ ohms. Neglecting any ohmic losses in the antenna structure itself, the radiation effeiciency of the antenna and coil combination is

Rad. Eff. $= 1.8 \times 10^{-2}/25.2 = .7 \times 10^{-3}$, or the efficiency is about .07% . If we could design a coil having a Q of 1000, the efficiency would be about .7%. This is one of the practical problems of small antennas.

Next consider the voltage to ground, or the voltage across the tuning coil: $|V|_{coil} = I X_c, |I| = (2 P_{rad}/R_r)^{\frac{1}{2}}$ so $|V| = 26600 (P_{rad})^{\frac{1}{2}}$ volts peak. This means if we radiate 100 watts, the voltages involved would be about 266 kV and of course even with 1% efficiency, the input power would be 10,000 watts, with most of this being dissipated in the coil. To radiate 10,000 watts would require megawatts of input and the voltages involved would be 2.6 megavolts. These are some of the practical problems of small antennas.

Fortunately for some applications, the radiated power needs to be only a few watts.

Problem 2.6

Suppose the ground screen consists of 120 radials and is circular with a 700 meter diameter: $b/\lambda = 350/3000 = .117$. The antenna height is $h/\lambda = 150/3000 = .05$. In this case, from the curves of Fig. 2.12, the resistance contributed by the imperfect ground screen system is about .8 ohms, call it .9 ohms to be conservative.

Consider an umbrella loaded monopole with 6 struts, angle 49°, so that the data in Figs 2.10 and 2.11 may be employed to estimate values. To try to keep the radiation resistance up, at the same time not letting the reactance get too high, let us consider an umbrella design with L/h = 1 . From the curves, the input resistance over a perfect ground would then be 1.5 ohms while the reactance could be expected to be about -210 ohms.

In this case a tuning coil would be desirable. Of course it should have a reactance at 100 kHz of +210 ohms. With careful design, the Q of such a coil could be 400 to 500; if we call it 420, the coil resistance implied is R = .5 ohms. The total input resistance of the coil-antenna combinationwould then be $R_{total} = 1.5 + .9 + .5 = 2.9$ ohms. The radiation efficiency of the antenna-coil combination would then be about 50%.

If the antenna were operating over a perfect ground, the power density at 20° from the horizon would be

$$PD_{\theta=70^\circ} = \frac{E_\theta^2}{120\pi}\Big]_{\theta=70^\circ} = G_d\, P_{rad}/4\pi r^2 = 3 \sin^2 70^\circ\, P_{rad}/4\pi r^2.$$

However, the ground screen is not large enough to alter the pattern that would be observed over an imperfect ground without a ground screen. The electric field strength will be only $\frac{1}{2}(1 + \rho^{H\perp}(\theta))$ times the value over a perfect ground. The values are

$$1 + \rho^{H\perp}(70^\circ) = \frac{2(-j100)^{\frac{1}{2}} \cdot .34}{(-j100)^{\frac{1}{2}} \cdot .34 + 1}; \quad \left|1 + \rho^{H\perp}(70^\circ)\right| = \frac{6.8}{4.19} = 1.625$$

so that in the installation proposed $|E|_{actual} = .815 |E|_{per.\,gd}$

To account for the effect of the imperfect ground on the pattern, we should make the calculation as though the required field over a perfect ground should be 100/.815 mv/m

Prob. 2.6 cont.
rather than 100 mv/m.

If the antenna were operating over a perfect ground, the power density at 20° from the horizon would be $PD = 3 \sin^2 70^\circ P_{rad}/4\pi r^2$. Thus the power required to produce the required rms field at 1000 miles is

$$P_{rad} = \frac{|E_\theta|^2 \times (1.609)^2 \times 10^{12}}{90 \times .882} = \left(\frac{.1}{.815}\right)^2 \times .326 \times 10^{11}$$

$$= 490 \text{ megawatts}.$$

With a radiation efficiency of 50%, this means that the power input required would be about 980 megawatts, which is a little discouraging.

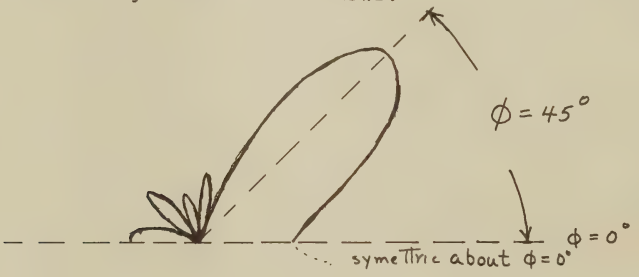
The input current would be $|I| = (10^9/3)^{\frac{1}{2}} = 18200$ amps. The voltage to ground and across the coil would be $|V| = 18.2 \times 2.10 \times 10^5 = 3.8 \times 10^6$ volts.

From these numbers we see that we had better be prepared to argue against a field strength requirement of as high as 100 mv/meter at 1000 miles.

Problem 3.1

The array factor for a five element array will be given by the $\sin\frac{5}{2}\Psi / \sin\frac{1}{2}\Psi$ curve given in Fig. 3.8a. To have the maximum at 45° to the line of the array, the phase shift will be $\alpha = -\beta d\,(.707)$, element to element. To decide on βd (the element spacing in wavelengths), we can say that in order to obtain as narrow a main lobe as possible, we want βd to be large; however, we must not make it so large that we get grating lobes or spurious lobes. Examination of the graphical constructions illustrated in Fig. 3.8a shows that the visible range will extend to a value $\Psi = +\alpha - \beta d = -\beta d(1 + .707)$. Further examination of the $\sin\frac{5}{2}\Psi/\sin\frac{1}{2}\Psi$ curve shows that if we let the visible range extend to about $\Psi = -1.67\pi$, we will avoid a grating lobe but will permit a back lobe of about the same size as the other side lobes. To proceed, we make this decision, and this determines βd: $-\beta d(1.707) = -1.67\pi$, or $\beta d \doteq .98\pi$. Then $\alpha = -.98\pi(.707) = -.69\pi$.

With these parameters selected, we are now ready to carry out the construction to give the array factor. Using an overlay on the $\sin\frac{5}{2}\Psi/\sin\frac{1}{2}\Psi$ curves, we obtain an array factor as follows:



We have not yet specified whether the array is to be a collinear or a non-collinear type. (Currents are z-directed)
a) If it is non-collinear, the array factor is the radiation pattern in the H-plane. In that E-plane which also contains the line of the array, the pattern is the array factor times $\cos\phi$ ($\cos\phi = \sin\theta$). In that E-plane which is also the $\phi = \frac{1}{2}\pi$ plane, the pattern is $\sin\theta$, with very small amplitude because this cut passes through the first side lobe.
b) If the array is collinear, the E-plane pattern is the AF times $\sin\theta$. The H-plane patterns are all circles.

Problem 3.2

We will decide that the continuous linear array to be designed is made up of non-collinear elements, currents along z, and that the specifications apply to the H-plane ($\theta = \frac{1}{2}\pi$) pattern. In such a plane, the radiation pattern is the same as the array factor.

The array factor for a sine tapered array is shown in Fig. 3.12. To place the main beam at 30°, we should have

$$\Psi_1 = \tfrac{1}{2}\beta L(\cos 30^\circ + k/\beta) = 0 \quad , \text{ or } k/\beta = -.866 \qquad (I)$$

Next we can look on the graph for the half power points. We find them (AF = .707) at $\Psi_1 = \pm .6\pi$. Thus, we solve for the angles on either side of 30° such that

$$\Psi_{1h.p.} = .6\pi = \tfrac{1}{2}\beta L(\cos\phi_1 - .866) \qquad (II)$$

$$\Psi_{2h.p.} = -.6\pi = \tfrac{1}{2}\beta L(\cos\phi_1 - .866) \qquad (III)$$

$\phi_1 = \phi_2 - 10^\circ$ so $-\cos\phi_1 + .866 = \cos(\phi_2 - 10^\circ) - .866$

or $\cos\phi_2 + \cos(\phi_2 - 10^\circ) = 1.732$

Using a trigonometric identity for the sum of two cosines

$$2\cos\tfrac{1}{2}(2\phi_2 - 10^\circ)\cdot\cos\tfrac{1}{2}(10^\circ) = 1.732, \quad \cos(\phi_2 - 5^\circ) = \frac{.866}{\cos 5^\circ} = .87.$$

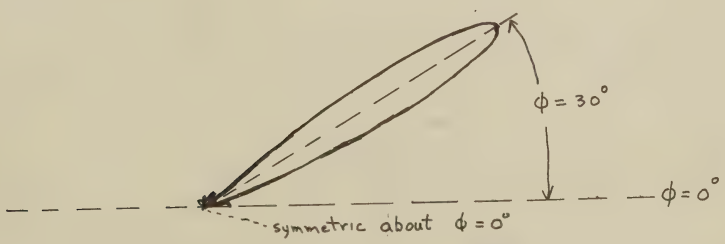
$\phi_2 - 5^\circ \doteq 29\frac{1}{2}^\circ$, so $\phi_2 = 34.5^\circ$, $\phi_1 = 24.5^\circ$, $\cos\phi_1 = .91$

So from (II), $\frac{1}{2}\beta L = \dfrac{.6\pi}{.91 - .866} = 13.55\pi$

From (I) $k/\beta = -.866 = \frac{1}{2}kL/\frac{1}{2}\beta L$, or $\frac{1}{2}kL \doteq -.866 \cdot 13.55\pi \doteq 11.75\pi$

Thus the array should be 13.55 wavelengths long and should have a phase constant of .866 that of space, or a phase velocity of c/.866.

The H-plane pattern would look as follows:



Problem 3.3.

Broadside array is to have $\lambda/4$ element spacing. Phase shift constant is to be zero. The visible range is from $\psi = -\pi/2$ to $\psi = \pi/2$. A five element array has 4 zeros. Suppose we try to adapt Schelkanoff's end fire design technique of spacing the zeros in the visible range. After a few trials, it becomes clear that the equal spacing of the zeros in the visible range results in "side lobes" that have greater amplitude than the main lobe. We conclude that spacing the zeros equally is not a general (magic) procedure.

In this case, to keep the side lobes down, we abandon the idea of equal spacing but keep the zeros in the visible range. Consider a design with the zeros at $\psi = \pm\frac{\pi}{4}, \pm\frac{\pi}{2}$

$AF(0°) = (14 \times 26)^2 = 13.2^2$

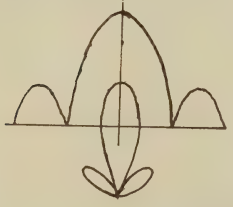
$AF(67\tfrac{1}{2}°) = (6.5)^2 \times 30 \times 35.5 = 4.5$ } ratio .34

$$AF = (z - e^{j\frac{\pi}{4}})(z - e^{-j\frac{\pi}{4}})(z - e^{j\frac{\pi}{2}})(z - e^{-j\frac{\pi}{2}})$$

$$= (z^2 - z \cdot 2 \cdot .707 + 1)(z^2 - 0 + 1)$$

$$= z^4 - 1.414 z^3 + z z^2 - 1.414 z + 1$$

The coefficients in this array polynomial give the relative current excitations at the elements.



The pattern has nulls at $\phi = 60°$ and $120°$ and has the general appearance indicated with a width between nulls of 60°.

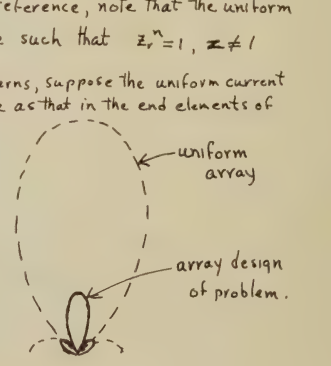
Contrast this with the uniformly excited broadside array with the same spacing. Referring back to Fig. 3.8a, we see that the uniform array has nulls at $.4\pi = \frac{\pi}{2}\cos\phi$, $\cos\phi = .8$ or $\phi_n = 37°$ and $143°$, or a width between nulls of 106°.

The side lobe to main lobe ratio is .2/1. For reference, note that the uniform array is given by $AF = \frac{z^n - 1}{z - 1}$ so the roots are such that $z_r^n = 1$, $z \neq 1$

or $z^5 = e^{\pm j 2n\pi}$, $z = e^{\pm\frac{2n\pi}{5}}$.

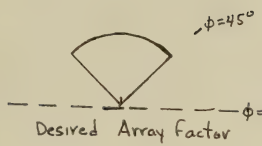
Distribution of zeros, uniform array

To compare patterns, suppose the uniform current level is the same as that in the end elements of the array above.



Patterns with amplitudes to same scale.

Problem 3.4



$$\Psi = \beta d \cos\phi$$

ϕ	Ψ	AF
0°	βd	0
30°	$.866\beta d$	0
45°	$.707\beta d$	C_0
60°	$.5\beta d$	C_0
90°	0	C_0
120°	$-.5\beta d$	C_0
135°	$-.707\beta d$	C_0
150°	$-.866\beta d$	0
180°	$-\beta d$	0

In order that the Array Factor as a function of Ψ be specified over a range of 2π, and be thus determined completely, we may take the spacing to be $\lambda/2$ so that $\beta d = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$.

Thus $AF(\Psi) = C_0 \quad -.707\pi < \Psi < .707\pi$

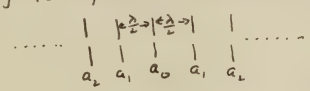
$= 0$ elsewhere

This array factor may be written in the form

$$AF(\Psi) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos n\Psi$$

$$a_n = \frac{1}{\pi}\int_{-.707\pi}^{.707\pi} C_0 \cos n\Psi \, d\Psi \,, \quad a_n = \frac{1}{n\pi}\sin n\Psi\Big]_{-.707\pi}^{.707\pi} = \frac{2}{n\pi}\sin n\,.707\pi \,.$$

The numbers a_n give the amplitudes of the currents in the array, a_0 being the center element, a_1 being the amplitudes in the elements on either side of center, and so on:



The next question is "how many elements are required?" To answer this question, we must somehow specify the accuracy of the approximation. If for example, we want the difference between the actual pattern and the desired pattern to be less than 5% of the peak amplitude, then, for example,

$$\left|\frac{\sin N\pi\times.707}{N\pi}\right| < \left|\frac{1}{20}\left[\frac{1.4}{2} + \sum_{n=1}^{N}\frac{\sin n\pi .707}{n\pi}\right]\right|$$

Trying out some numbers, it would appear that N $n = 7$ would be sufficient; this implies 15 elements in the array.

Problem 3.5

This problem is worked just as the example in the foregoing section. If for example, we take an array of 4 wavelengths, the sample points are $\phi_s = \cos^{-1}(\pm\frac{1}{4}n)$

$$a_n L = AF(\phi_s) = \sin^2\phi_s = 1-\cos^2\phi_s = 1-(n/4)^2, \quad n = 0, \pm1, \pm2, \pm3, \pm4$$

Thus $$I(x) = \sum_{-4}^{4} \frac{1}{4}\left(1-\left(\frac{n}{4}\right)^2\right) e^{j\frac{\pi}{2}nx}\left[U(x+2)-U(x-2)\right]$$

$$= \frac{1}{4}\left[1 + 2\sum_{n=1}^{4}\left(1-\left(\frac{n}{4}\right)^2\right)\cos\frac{\pi}{2}nx\right]\left[U(x+2)-U(x-2)\right]$$

The symmetry in the current distribution follows from the symmetry in the desired array factor. A plot of this function reveals that a 4 wavelength aperture is not required to produce an acceptable approximation to the desired array factor.

To make a detailed comparison between the synthesized array factor and the deired one, a fair amount of numerical calculation must be performed. In particular, we must evaluate the series

$$AF(\Psi) = L\sum_{n}' a_n \frac{\sin\left(\frac{\beta L}{2}\cos\phi + n\phi\right)}{\left(\frac{\beta L}{2}\cos\phi + n\phi\right)}$$

for different values of ϕ . This can be done with a simple digital computer program. For the convenience of the reader, such a program, written in FORTRAN Language, follows. The variable $\Psi(N) = \frac{1}{2}\beta L\cos\phi + n\pi = \pi\left(n + \frac{L}{\lambda}\cos\phi\right)$

```
      DIMENSION A(99),PSI(99),AF(99)
    1 FORMAT (I2)
    2 FORMAT (F20.8)
    3 FORMAT (11X,3HPHI,17X,2HAF,17X,5HAFACT//)
      READ (5,1) NTOTAL
      READ (5,2) AL
      READ (5,2) PHIMIN, PHIMAX, DPHI
      READ (5,2) (A(N), N=1, NTOTAL)
      WRITE (6,3)
      PI = 3.14159265
      PHIMIN = PHIMIN * PI/180.
      PHIMAX = PHIMAX * PI/180.
      DPHI = DPHI * PI/180.
      PHI = PHIMIN
   10 N=1
      AN = -(NTOTAL-1)/2
   15 PSI(N) = PI * (AN + AL * COS(PHI))          (Form Ψ(N))
      IF (PSI(N)) 20, 25, 20
   20 AF(N) = A(N) * SIN(PSI(N))/PSI(N)          (Form the terms in
      GO TO 30                                    the series)
```

Problem 3.5 cont'

```
 25  AF(N) = A(N)
 30  N = N+1
     AN = AN + 1.
     IF (N - NTOTAL) 15, 15, 35
 35  AFT = 0.
     DO 40 N = 1, NTOTAL
 40  AFT = AFT + AF(N)                    (Sum the series)
     PHIDEG = PHI * 180.0/PI
     AFACT = SIN(PHI) * SIN(PHI)
     WRITE (6,4) PHIDEG, AFT, AFACT
     PHI = PHI + DPHI
     IF (PHI-PHIMAX) 10, 10, 50
 50  CONTINUE
     STOP
     END
$ DATA
 9
       4.0
       0.0
     180.1
       5.0
       0.0
       0.4376
       0.7500
       0.9375
       1.0000
       0.9375
       0.7500
       0.4376
       0.0
```

(NTOTAL - total no. of sample points)

(AL - length of array in wavelengths)

(PHIMIN) } range of angle ϕ in calculation
(PHIMAX) }

(DPHI) Increment in angle ϕ

(A(1))
(A(2))
⋮
Sample points A(N)

Problem 3.6

A seven element array implies a design based on the Chebyshev polynomial of order 3, $T_3(x) = 4x^3 - 3x$.

$x = a \cos\psi + b$ and $AF = A_o + \sum_{n=1}^{3} 2 A_n \cos n\psi$

where $A_o = 6a^2b + 4b^3 - 3b$

$2A_1 = 3a^3 + 12ab^2 - 3a$

$2A_2 = 6a^2b$

$2A_3 = a^3$

We are to radiate broadside with a beamwidth between nulls of 30° . The angle to the first null is then 15°. The spacing is not specified in the problem so we may set it at $\lambda/2$ between elements or $\beta d = \pi$. In this case then $b - a = -1$. The root of interest lies at $x = \cos \pi/6 = .866$ so the second equation for a and b is

$$.866 = a \cos(\pi \cos 75^\circ) + b$$

From these two equations we find $a = 1.108$, $b = .108$ so $x_o = a + b = 1.216$, from which $R = 4(1.216)^3 - 3(1.216)$ or $R = 3.54$. That is, the main lobe to side lobe level is 3.54 or the side lobe level is about -11 db. (The beam is somewhat narrower than in the example in the text, so as would be expected, the side lobe level is higher.)

A sketch of the pattern, based on a construction similar to that in Fig. 3.26, is shown at left.

Next consider the pattern at a new frequency

$$AF = A_o + \sum_{n=1}^{3} 2 A_n \cos n\psi$$

It is stated that the excitation is independent of frequency; this implies that the A_n's in the array factor remain the same. But $\psi = \beta d \cos\phi$ so since at f_o, $\beta d = \pi$, then at $f_o + .1f_o$, $\beta d = 1.1\pi$, and at $f_o - .1f_o$, $\beta d = .9\pi$. To find the pattern at the higher and lower frequency, it is only necessary to perform the last step in the transformation process. $AF(\psi)$ does not change with frequency if the A_n's do not change. Thus, only the construction relating $AF(\ )$ to the visible angle ϕ must be done. This means that all we need is a circle of radius $.9\pi$ and one of radius 1.1π , in conjunction with the

Prob. 3.6 cont'.
AF(Ψ) curve. Note that the relative side lobe level is the same at all three frequencies. However, the beamwidth between nulls is different. A quick calculation shows that at $.9f_o$ the beamwidth is 33^o while at $1.1f_o$, it is about 25^o .

Problem 3.7

A seven element Chebyshev array may be designed on the basis of a 3^{rd} order Chebyshev polynomial $T_3(x) = 4x^3 - 3x$. The side lobe level is determined by the value of x_o which corresponds to the beam maximum. For a side lobe level of - 20 db, $x_o = 1.541$ as in the example in the text.

For the optimum endfire pattern, the phase shift, α , as well as the constants a and b must be determined. Let us set the spacing at $d = \frac{1}{4}\lambda$, $\beta d = \frac{1}{2}\pi$. Then according to equations 37, 38 and 39 in the text

$$a = \frac{1.541 + 3 + 0}{2} = \frac{4.541}{2} = 2.2705$$

$$b = 1 - 4.541/2 = -1.2705$$

$$\sin \alpha = \frac{1.541 - 1}{2.\ \frac{4.541}{2}} = .1192, \quad \alpha = 6^o50' \text{ or } .1195$$

As in the example in the text, the amplitudes of the coefficients are given by

$$A_o = 6a^2b + 4b^3 - 3b$$
$$2A_1 = 3a^3 + 12ab^2 - 3a$$
$$2A_2 = 6a^2b$$
$$2A_3 = a^3$$

- 43
+ 36
-14.6
+5.8

The element to element phase shift is $6^o50'$ or .1192 radians.

The array factor is the same as that labeled Duhamel's Optimum in Fig. 3.20 .

Problem 3.8

In this unequally spaced array, first check to see if the quantity $j\Psi\Delta_n/d$ is small. With the stated phase shift of 90° and with errors relative to an unequally spaced array with quarter wavelength spacing, the interesting range of is 0 to $-\pi$. As a worst case take $\Psi = \pi$, and n = 5. With this worst case the quantity has magnitude about .5. Thus over most of the range, a good approximation is

$$e^{j\Psi\Delta_n/d} - 1 = j\Psi\Delta_n/d$$

Thus the difference between the array factor with unequal spacing and that with equal spacing is

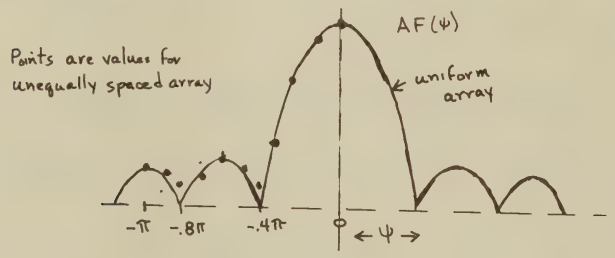
$$AF(\Psi) - AF_e(\Psi) = \sum_{n=0}^{N-1} e^{jn\Psi} j\Psi (.95)^{-n} \frac{\lambda}{32d}$$

$$= j\frac{\Psi\lambda}{32d} \sum_{n=0}^{N-1} e^{jn\Psi} \frac{1}{(.95)^n} = j\frac{\Psi}{8} \sum_{n=0}^{N-1} \left(\frac{e^{j\Psi}}{.95}\right)^n$$

The summation is a geometric progression with a common factor of $e^{j\Psi}/.95$. The sum of terms is then

$$\sum_{n=0}^{4} e^{jn\Psi} \cdot \frac{1}{(.95)^n} = \frac{1 - \frac{e^{j5\Psi}}{.95^5}}{1 - \frac{e^{j\Psi}}{.95}} = \frac{1 - 1.29e^{j5\Psi}}{1 - 1.05e^{j\Psi}}$$

With a digital computer, it is easy to sum the exact expression for the array factor of the unequally spaced array. The sketch below shows the array factor for the equally spaced array with points for the unequally spaced array superposed.



Problem 3.9

Let us design a retrodirective array of the type indicated in Fig. 3.33. The phase of the signal from each element is conjugated by a mixing process to obtain the retrodirective feature, as explained in the text. The required amplitudes can then be obtained by adjusting the relative gains of the amplifiers. Care should be taken in the selection of the elements in the array so that the directly scattered field is small.

To make the reradiated array factor the desired optimum Chebyshev in the broadside direction, we can take the element spacing to be a half-wavelength, so that $\beta d = \pi$. We can base the design of the five element array on a Chebyshev polynomial of order 2, $T_2(x) = 2x^2 - 1$. To obtain a side lobe level of - 20 db, we let the pattern extend to an x_o such that $T_2(x_o) = 10$ or $x_o^2 = 5.5$ $x_o = 2.345$. The quantity x and ψ are related by $x = a \cos \psi + b$, where $a = \frac{1}{2}(x_o+1) = \frac{1}{2}(3.345)$, $b = \frac{1}{2}(x_o-1) = \frac{1}{2}(1.345)$.

The current excitation coefficients are then $A_o = a^2 + 2b^2 - 1 = 2.70$, $A_1 = 2.25$, $A_2 = 1.38$. Thus, relative to the gain of the outside element amplifiers, the amplifier for the center element should have a gain of 2.08, and those of the interior pair of elements should each have a gain of 1.63.

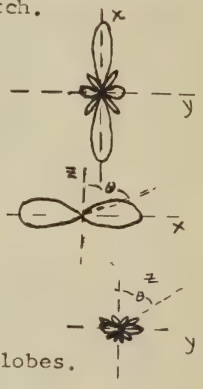
Since we have taken a spacing $\beta d = \pi$, so that the visible range includes the value π , the equations for a and b for the "off broadside" beams are the same as the broadside case. Thus our design gives Chebshev beams in the range $\phi = 30^\circ$ to 150° as specified.

Problem 4.1

A full wave dipole has a pattern given by $E_\theta \sim \frac{\cos(\pi\cos\theta) + 1}{\sin\theta}$. This element pattern is sketched in Fig. 4.2 of the text.

Suppose the currents flow in the z-direction, and the the elements are lined up along the y-axis.

a) Each element has the same amplitude and phase. The array factor as a function of Ψ is plotted in Fig. 3.8a. The array factor as a function of the angle with respect to the line of the array can be determined by the projection of the $AF(\Psi)$ curve on a circle of radius $\beta d = \pi$, whose center is at $\Psi = 0$, as discussed in connection with Figs. 3.7 and 3.8 of the text. The array factor has an appearance as indicated in the next sketch. Thus

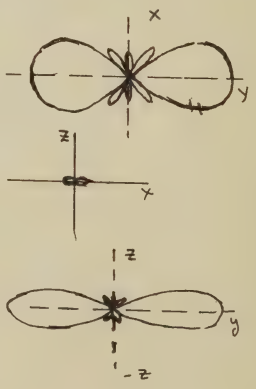


i) in the xy-plane, the pattern is determined by the array factor, since the element pattern is omnidirectional in this pane.

ii) in the xz-plane, the pattern is determined by the element patter n.

iii) in the yz-plane,the pattern is the product of the array factor and the element pattern. Thus, in this plane, the pattern is mostly side lobes, or small lobes.

b) The elements all have the same amplitude, but are all phased for end fire operation, $\alpha = -\beta d$. The array factor as a function of the angle with respect to the line of the array may be sketched as in Fig 3.8a , the circle has radius $\beta d = \pi$, with center at $\Psi = -\pi$.



i) The pattern in the xy plane is determined by the array factor.

ii) In the xz-plane, the pattern is determined by the element pattern. The field strength is very small, however, since the"cut" is through a side lobe of the array factor.

iii) In the yz-plane, the pattern is the product of the element pattern and the array factor.

Problem 4.2

The monopole has an H/D ratio of 100. From Figs. 4.10 a and b , the impedance values can be found. These can be converted to admittances values with the aid of a Smith Chart (180° rotation on chart).

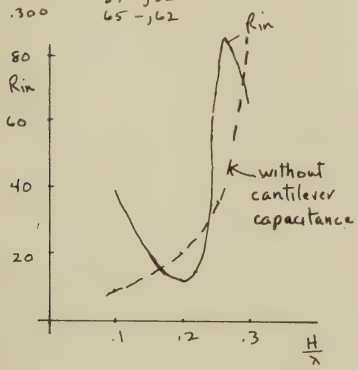
To a first approximation, the cantilever type of support introduces a shunting admittance given by

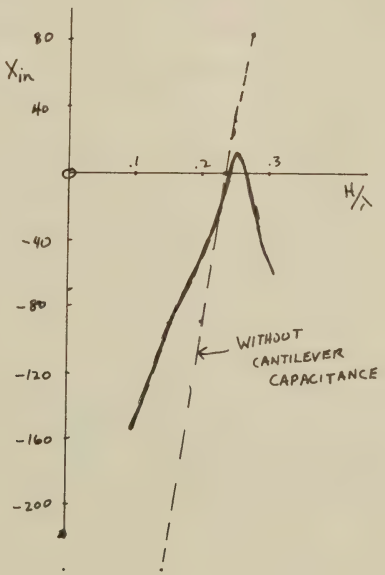
$$Y = \frac{j\omega\epsilon\, 2\pi}{\ln \frac{r_o}{r_c}} \times 12 \times .0254 \quad \text{where } \frac{r_o}{r_c} = \frac{4}{3} \text{ so } \ln\frac{4}{3} = .287,\ \omega = 2\pi f = \frac{2\pi}{\sqrt{\mu_o\epsilon_o}}\frac{1}{\lambda}$$

$$Y = j\,2\pi \frac{1}{\sqrt{\mu_o\epsilon_o}} \frac{\epsilon_o\epsilon_r}{25} \frac{H}{\lambda} \frac{2\pi}{.287} = j\frac{1}{60}(3-j1)\frac{H}{\lambda}\frac{2\pi}{7.18} = (.0145 + j.0434)\frac{H}{\lambda}$$

This shunting admittance is added to the monopole admittance for each value of H/λ . The total admittance values so obtained can be inverted with the aid of the Smith chart. The input impedance found in this way is displayed in the table and graphs below. The ideal (unshunted) monopole impedance is sketched for comparison. Clearly the cantilever type of support has a rather drastic effect on the input impedance. The increasing resistance at the lower frequencies (smaller H/λ) is characteristic of the impedance behavior when there is capacitance shunting the input of a monopole or dipole antenna.

H/λ	Z
.1	38 - j156
.125	26 - j116
.15	20 - j91
.175	16 - j68
.20	14 - j46
.225	16 - j25
.25	55 + j12.5
.275	84 - j32
.300	65 - j62





$\theta_2 = \pi/2$

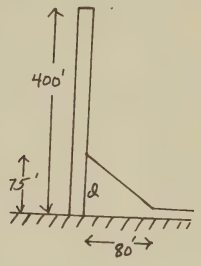
$\theta_1 = \frac{2}{400} = .005$

$\theta/2 = .0025$

Problem 4.3

An approximate solution may be obtained either with a biconical model or a cylindrical model.

a) Biconical Model



Cone Angle $\doteq \frac{s}{r} = \frac{4}{200} = .02$ rdns

Half Angle $\theta_1 \doteq .01$

$Z_o = 120 \ln \cot \frac{\theta_1}{2}$

$\doteq 120 \ln \frac{4}{\theta_1} \doteq 120 \ln \frac{400}{200}$

$\doteq$ ~~636 Ω~~ 71.9 Ω

At 550 kHz, $\frac{L}{\lambda} = \frac{H}{\lambda} = \frac{400 \times .55}{984} = .224$, $\frac{L-d}{\lambda} = .182$

$\frac{d}{\lambda} = \frac{75 \times .55}{984} = .0420$, $\beta d = 2\pi \times .0420$

From Fig 4.8, $Z_a = 50 + j140 = (636)^2 Y_t$

so $\frac{Y_t}{Y_o} \doteq \frac{50 + j140}{636} \doteq$ ~~.079 + j.22~~ .069 + j.195

Using the Smith Chart to transform the "short" through .042λ 15° and Y_t/Y_o through .182λ 65° we obtain

$\frac{Y_{in}}{Y_o} = -j3.7 + .52 + j.405 =$ ~~1.52 + j.35~~ 1.4 + j0

or $Z/Z_o =$ ~~.63 − j.15~~ .714, $Z_{rap} =$ ~~400 − j95~~ 513 ohms dipole

$=$ ~~260 − j63~~ monopole 256 Ω real

This is the impedance at a point 75 feet above the ground. We must now add the inductance of the wire to ground. This can be estimated by transforming along a non-uniform transmission line. Assuming an average characteristic impedance of about 1000 Ω, we transfer $Z_{rap}/1000$ through a distance of $80 \times .55/984 = .0446\lambda$ which gives 16°

$\frac{Z_{in}}{1000} \doteq .21 + j.24$ or $Z_{in} \doteq$ ~~210 + j240~~ 250 + j270.

b) Cylindrical model

From Figs. 4.10 a & b, for length $\frac{L-d}{\lambda} = .182$, $Z_{in} = 20 - j75$

Referring to Chap 5, pp. 217 and 219 for an approximation to the inductance, we find that the impedance of the 75 foot mast below the

Prob. 4.3 cont.'

Feed point should be

$$Z_{in} \doteq 1600(.042)^2 + j60 \ln \frac{2}{\frac{1.781 \cdot 2\pi \cdot 3.55}{984}} \times 2\pi \times .042 + \left(\frac{2\pi \times .55 \times 10^6 \times 4\pi \times 10^{-7}}{2 \times 4 \times 10^7}\right)^{\frac{1}{2}} (1+j)\frac{208}{2\pi}$$

$$\doteq 3 + j81$$

$$Z_{Tap} \doteq \frac{1}{Y_{in_1} + Y_{in_2}} \doteq 250 - j10$$

Accounting for the lead-in inductance gives

$$Z_{in} \doteq 270 + j260.$$

Problem 4.4

First, determine the characteristic impedances. For the balanced pair over ground (from a reference such as ITT Ref. Data for Radio Engnrs., 4th Ed., pg.590)

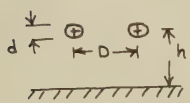
$$Z_o = \frac{276}{\sqrt{\epsilon}} \log_{10}\left[\frac{2D}{d} \frac{1}{\sqrt{1+(D/2h)^2}}\right] \qquad d \ll D, h$$

This is stated to be 300 ohms, so with h/D = 4, we can solve for d to find d = .49" .

For the double wire (wires in parallel) over ground ($d \ll D, h$)

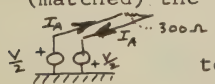
$$Z_o = \frac{69}{\sqrt{\epsilon}} \log_{10} \frac{4h}{d} \sqrt{1+\left(\frac{2h}{D}\right)^2}$$

$$\doteq 200\,\Omega$$

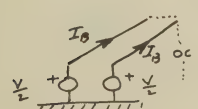


To solve, replace the unbalanced excitation by push-push and push pull sources, as indicated below.

On the double wire line over ground, the open circuit transformed through an eighth of a wavelength gives $Z_{in} = -j\, Z_o \cot \beta L = -j\, 200$. On the balanced line (matched) the input impedance is 300 ohms.



The voltage at the coax output is measured to be 100 V. The currents I_1 and I_2 in the two sides of the line are



$$I_1 = I_A + I_B = \frac{100}{300} + \tfrac{1}{2}\, V/2Z_{in} = \frac{1}{3} + j\,\frac{1}{8}$$

$$I_2 = I_A - I_B = \frac{1}{3} - j\,\frac{1}{8}$$

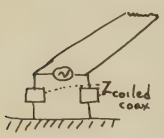
Prob. 4.4 cont.

The effective impedance terminating the coax is

$Z_T = V/I_1 = 100/(\frac{1}{3} + j\,\frac{1}{8}) = 263 - j\,99$

The normalized impedance terminating the coax is then $Z_T/Z_o = 3.5 - j\,1.32$, which gives a VSWR = 4.

Consider the design of a coiled coax balun. The schematic excitation with balun in place is as indicated.



To keep currents from flowing to ground, the impedance of the coiled coax balun should be high compared to 300 ohms. With the 75 ohm coax such as RG 11A/U we could conveniently form a 10 turn coil having a diameter of 5" and a coil length of as much as 10" (5" each side). Using single layer solenoid formaulas as an approximation L = 100 x 5 x .0273 = 13.6 microhenries. The reactance at mid-band would then be X = 8500 ohms. Thus, the current to ground would be .035 that of the current to the load. This would probably be acceptable. The input impedance would be

$$Z_{in} = \frac{300 \times j8500}{300 + j8500} = 300 + j\,11.5 \text{ ohms}$$

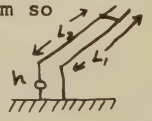
A four to one impedance transforming balun could be designed as described in the text, sections 4.5.1,4.5.2.

Problem 4.5

The input impedance of a folded monopole is

$$Z_{in} = \frac{4\,Z_A Z_I}{4\,Z_A + Z_I}$$ where Z_A is the antenna impedance and Z_I is the input impedance to the balanced line. We can make Z_I essentially infinite by introducing a shorting bar as shown and making $L_2 + h$ equal to a quarter wavelength. Then $Z_{in} = 4Z_A$.

We want this to be $50 + j\,0$ ohms. The line itself is nearly lossless so the input resistance must come from radiation resistance. We will try to make the current distribution in the vertical section nearly uniform so that $R_{rad} = 1600\,(h/\lambda)^2$ or $\frac{h}{\lambda} = .088$ and $L_2/\lambda = .25 - .088 = .162$.



Prob.4.5 cont.

The length L_1 of the double wire above ground must be adjusted to produce the essentially uniform current and to provide for an effective antenna reactance of zero. The length L_1 will be in the vicinity of a quarter wavelength, but the exact length for tuning will depend on some details.

Suppose for example that the frequency of operation is such that h turns out to be 4" (recall $h/\lambda = .088$) In this event, the characteristic impedance of the double wire above ground (the push-push loading transmission line for the vertical pair) is

$$Z_o = \frac{69}{\sqrt{\epsilon}} \log_{10} \frac{4h}{d} \sqrt{1+\left(\frac{2h}{D}\right)^2} \doteq 138$$

If L_1 were exactly $\frac{1}{4}\lambda$, the open circuit at the end would transfer into a short at the top of the vertical wires. The antenna input impedance, Z_A, would be approximately $Z_A = R_r + j X_L$, where X_L is the inductive reactance associated with the vertical members. The latter can be roughly estimated by $X_L \doteq Z_o' \tan 2\pi h/\lambda \doteq .6 Z_o'$ where Z_o' is the average characteristic impedance of the vertical pair. Taking 100 ohms as an approximation for Z_o', we have $X_L = 60$ ohms. Thus, L_1 should be shorter than $.25\lambda$ to cancel the inductance. Since $60/138 = .435$, from the Smith Chart it follows that L_1/λ should be about .185.

Problem 4.6

Assuming that the current elements are very short and have uniform currents along their lengths with unity input currents, the mutual impedance expression is

$$Z_{12} = -\int E_{ao} \cdot J_{ib}\, dV = E_{ao,z}\, \ell_2$$

where ℓ_2 is the very short length of antenna b and $E_{a,o,z}$ is the z-component of the field produced by antenna a at the position of b (but with antenna b removed). In the collinear arrangement, $E_{ao,z}$ is just the radial component of the electric field of an elemental current

$$E_{ao,z} = \frac{\ell_1}{2\pi} e^{-j\beta r} \sqrt{\frac{\mu}{\epsilon}} \frac{1}{r^2}\left[1 + \frac{1}{j\beta r}\right]$$

Prob. 4.6 cont.

It follows that the mutual impedance is

$$Z_{12} = -240 \frac{\ell_1 \ell_2}{\lambda^2} e^{-j\pi} \left(1 + \frac{1}{j\pi}\right)$$

If the current distributions were triangular instead of uniform as assumed, the value would be approximately one-fourth of the value indicated.

To find the directive gain, we can compare the directive gain of the couplet to that of one of the elements by itself

$$\frac{G_{d,\,array}}{G_{d,\,single}} = \frac{PD_{array}}{PD_{single}} \times \frac{|I_{in}|^2 R_{r\,single}}{Power\ in,\ array}$$

$\frac{PD_{array,\,max}}{PD_{single,\,max}} = 4$ since the field is twice as great in the direction of the maximum, assuming the same input current in each case.

We will then calculate the relative directivity. If the elements are the same length, then the input impedance of the elements are the same, so if the currents are the same, we have

$$Z_{in} = Z_{11} + Z_{12}$$

$$Re(Zin) = R_{11} + R_{12} = 80\pi^2 (\ell/\lambda)^2 + 240 (\ell/\lambda)^2$$

The power input to the array is then

$$P_{in,\ array} = 2(\ell/\lambda)^2 \left(80\pi^2 + 240\right) I_{in}^2$$

$$P_{in,\ single} = I_{in}^2\, 80\pi^2 (\ell/\lambda)^2$$

or

$$\frac{G_{di,array}}{G_{di,\,single}} = 4 \cdot \frac{80\pi^2}{2[80\pi^2 + 240]} = 1.533, \quad \text{or} \quad G_{di,array} = 2.3$$

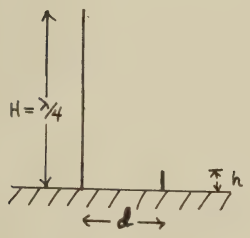
Note that the directivity is the same for either the uniform distribution of current or the triangular distribution.

Problem 4.7

The mutual impedance between two antennas is $Z_{12} = -\frac{1}{I_1 I_2} \int E_2 \cdot J_1 \, dv$. The integral can be taken either over the quarter wave monopole, or over the short vertical monopole. Clearly, it will be easier to take J_1 to be the short vertical; the integration will be much simpler since E_2 will be essentially constant over J_1 . The field of the quarter wave monopole is

$$E_{z2} = -\frac{j}{4\pi}\left(\frac{\mu}{\epsilon}\right)^{\frac{1}{2}} I_m \left(\frac{e^{-j\beta R_1}}{R_1} + \frac{e^{-j\beta R_2}}{R_2} \right) \quad \text{(eqn. 16)}$$

where in this case $R_1 = R_2 = (d^2 + (\frac{1}{2}\lambda)^2)^{\frac{1}{2}}$



Assuming a triangular current distribution on the short vertical, and unit currents I_1 and I_2

$$Z_{12} \doteq j\,60\,\frac{e^{-j\beta R_1}}{R_1}\,\frac{h}{2}$$

$$\beta R_1 = \sqrt{(\beta d)^2 + (\tfrac{\pi}{2})^2}$$

or

$$Z_{12} = \frac{30\beta h}{\sqrt{(\beta d)^2 + (\frac{\pi}{2})^2}} \left[\sin\sqrt{(\beta d)^2 + (\tfrac{\pi}{2})^2} + j \cos\sqrt{(\beta d)^2 + (\tfrac{\pi}{2})^2} \right]$$

Problem 4.8

It is stated that the antennas are to be driven so that $I_1 = j\,I_2$. The input voltages are therefore

$$V_1 = I_1 Z_{11} - j\,I_1 Z_{12}$$

$$V_2 = jI_2 Z_{12} + jI_2 Z_{22}$$

Here, $Z_{11} = Z_{22} \doteq Z_{self} \doteq 36 + j\,40$
$Z_{12} \doteq 20.5 - j\,14.5$ } from the mutual impedance curves

$V_1 = I_1[36 + j30 - j(20.5 - j14.5)] = I_1[21.5 + j9.5]$, $V_2 = I_2[50.5 + j50.5]$

The total input Power is $|I_{in1}|^2 R_{1in} + |I_{in2}|^2 R_{2in}$. $R_{1in} = Re\,\frac{V_1}{I_1}$, $R_{2in} = Re\,\frac{V_2}{I_2}$, $|I_{2in}|^2 = |I_{1in}|^2$

$P_{in} = 10 \times 10^3 = |I_{1in}|^2 [21.5 + 50.5]$, $|I_{1in}|^2 = 10^4/72$, $I_1 = \frac{10^2}{\sqrt{72}} = 11.8$ Amp Rms

$I_2 = -j\,I_1 = -j\,11.8$ Amp

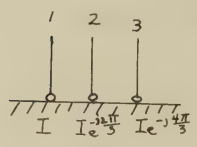
∴ The required driving voltages are

$V_1 = 11.8\,(21.5 + j9.5) = 253 + j112 = 276\angle 24^\circ$

$V_2 = -j11.8\,(50.5 + j50.5) = 594 - j594 = 840\angle -45^\circ$

Problem 4.9

First, find the input impedance to each tower



$$V_1 = I_1Z_{11} + I_2Z_{12} + I_3Z_{13}$$

$$V_2 = I_1Z_{12} + I_2Z_{22} + I_3Z_{23}$$

$$V_3 = I_1Z_{13} + I_2Z_{23} + I_3Z_{33}$$

where $Z_{11} \doteq Z_{22} \doteq Z_{33} \doteq 36 + j\,30$

$Z_{12} \doteq Z_{23} \doteq 20.5 - j\,14.5$

$Z_{13} \doteq -6 - j\,10$

$$V_1/I_1 = 36 + j30 + (I_2/I_1)(20.5-j14.5) + (I_3/I_1)(-6-j10)$$

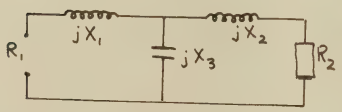
$$I_2/I_1 = e^{-j2\pi/3} \qquad I_3/I_1 = e^{-j4\pi/3}$$

so $Z_1 = V_1/I_1 = 35 + j\,19$

Similarly, $Z_2 = 16 + j\,45$

$Z_3 = 33 + j\,70$

The type of feeding and matching network that is appropriate depends on frequency. Let us suppose the frequency is in the megacycle region. In this case, lumped matching networks and impedance transformers can be built conveniently. For example, a T-network can be used both to bring about an impedance transformation and a desired phase shift as well (for reference, see Terman, or Jordan, or Jasik, op.cit.). If a phase lag ϕ is desired ($I_2 = kI_1e^{-j\phi}$), the values are



[For convenience, define $b = r_t \sin\phi$, $r_t = (R_1/R_2)^{\frac{1}{2}}$],

then $X_1 = R_1(1 - r_t\cos\phi)/b$

$X_2 = R_1(r_t - \cos\phi)/b$

$X_3 = -R_1/b$

We can plan to feed with transmission lines located near antenna 1 (ie., with the transmitter located near antenna 1). If we match all transmission lines, we can easily get a 90° phase shift by virtue of the lines leading to element 2 and 180° in those leading to element 3.

Prob.4.9 cont'.

The current amplitudes are all to be equal. This means that the ratios of powers which must be supplied to the three elements are $P_1 : P_2 : P_3 = 35 : 16 : 33$. We can plan to use 50 ohm lines. Thus we can plan to incorporate an impedance transformation at element 3 to provide a 50 ohm impedance level. If we employ a parallel branching at element 2, then to have a power split in the desired ratio, we must have $Z_2/Z_3 = 33/16$ or $Z_2 = 100$ ohms. The input impedance will then be 33 ohms at the junction, so we can plan to use the same type of matching network as at element 3 to get the impedance level back up to 50 ohms.

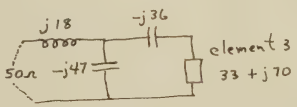
The parameters for the different elements are as follows:

Element	R_1/R_2	r_t	ϕ	$b = r_t \sin\phi$	$\cos\phi$	$r_t \cos\phi$
3	50/33	1.23	60°	1.07	½	.615
2	100/16	2.5	30°	1.25	.866	2.17
1	70/33	1.46	60°	1.26	½	.73

So, for element 3, $X_1 = 18$ ohms

$X_2 = 34$ ohms

$X_3 = -47$ ohms

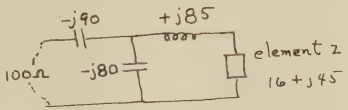


for element 2

$X_1 = -94$

$X_2 = 130$

$X_3 = -80$

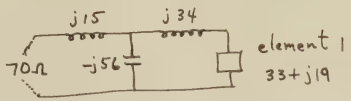


for element 1

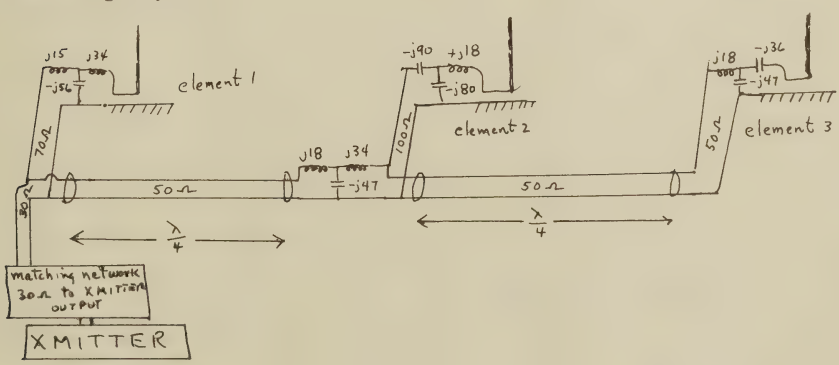
$X_1 = 15$

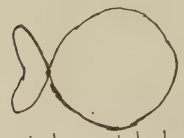
$X_2 = 53$

$X_3 = -55.5$



Feeding Layout:





Pattern in horizontal plane

In the horizontal plane, the radiation pattern is the array factor

$$AF = \frac{\sin\frac{3\psi}{2}}{\sin\psi/2}$$

$$\psi = \beta d\cos\phi + \alpha = \frac{\pi}{2}\cos\phi - \frac{2\pi}{3}$$

It should be possible to have designs such that the efficiency of the matching networks is 90% or better . With reasonably good antenna structures then the over all efficiency might be about 80%. This means that there could be about 800 watts available for radiation. Thus $|I|^2 = 800/84$, or we might arrange to have a current amplitude of about 3.1 A.rms . The field strength is given by

$$E_\theta = \frac{j60}{r} e^{-j\beta r}\, 3.1\left[\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}\right] \frac{\sin\frac{3}{2}(\frac{\pi}{2}-\frac{2\pi}{3})}{\sin\frac{1}{2}(\frac{\pi}{2}-\frac{2\pi}{3})}$$

or at one mile : $\theta = 90°$, $r = 1.609\,m$

$$E_\theta \doteq .3 \text{ volts/m rms}$$

Problem 4.10

From the definitions, assuming a lossless structure

$$G_{di} = \frac{PD(0,\frac{\pi}{2})}{P_{rad}} 4\pi r^2 = \frac{PD(0,\frac{\pi}{2})}{|I_{in}|^2 R_{in}} 4\pi r^2$$

When the dipole is located a quarter of a wavelength from the plane

$E_\theta = \frac{2\times j60}{r} I_{in}$ in the maximum direction, so $(\varphi=0, \theta=\frac{\pi}{2})$

$PD(0,\frac{\pi}{2}) = \frac{4\cdot 3600\,|I_{in}|^2}{120\pi r^2} = \frac{120\,|I_{in}|^2}{\pi r^2}$. Also $R_{in} = Z_{11} - Z_{12}$ so

From Fig. 4.37 $R_{in} = 80\,\Omega$. This gives $G_{di} = 5.59$.

At small spacings $E_\theta = \frac{j60 I_{in}}{r}[1 - e^{-2j\beta h}]$ so

$$P.D. = \frac{2\cdot 3600}{120\pi r^2}|I_{in}|^2[1-\cos 2\beta h] \text{ and}$$

$$G_{di} = \frac{240\,[1-\cos 2\beta h]}{R_{in}}$$

Prob.4.10 cont.

Assuming that the antenna and the reflecting plane are lossless, we can use this formula to compute the following table of values for the directivity as a function of spacing from the plane ($R_{in} = R_{11} - R_{12}$, the latter being obtained from the mutual impedance curves)

h/λ	0.2	0.15	0.125	0.1	0.05
G_{di}	6.24	6.96	7.2	7.56	7.69

Although the directivity increases with decreasing spacing, it should be noted that the absolute gain is decreased by the ohmic resistance in the structures. This latter effect is most pronounced at small spacings. (for example, add 1 ohm for ohmic resistance and compute the new gain at a spacing of .25 λ and the new gain at a spacing of .05 λ).

Problem 4.11

For a three element Yagi at 100 MHz, the lengths will be of the order of 1.5 m. Thus, the elements could be constructed of tubing of about 5/8" OD, in which case the length to diameter ratio would be about 100.

In the absence of other information, let us employ spacings and lengths which will work well in two element Yagi designs. We have a digital computer program which estimates impedances (both self and mutual) based on the equation on page 192 of the text. Thus we can try a couple of different designs without too much work.

Try #1 $\ell_{ref} = .26\lambda$, $\ell_{driv} = .24\lambda$, $\ell_{dir} = .225\lambda$

spacing ref. to driv. = $.15\lambda$, spacing dir. to driven = $.10\lambda$

director $Z_{11} = 54 - j23$; $Z_{12} = 55 + j2$

driven $Z_{22} = 64 + j14$; $Z_{13} = 37 - j26$

reflector $Z_{33} = 82 + j67$; $Z_{23} = 60 - j7$

$Z_{in} = 18 - j5$

$G_{di} = 8.5$ db (?)

Array factor shown dotted

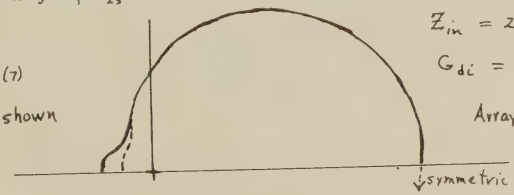
Try #2 $\ell_{ref} = .26\lambda$, $\ell_{driv} = .25\lambda$, $\ell_{dir} = .225\lambda$

spacings same

director $Z_{11} = 54 - j23$; $Z_{12} = 58 + j4$

driven $Z_{22} = 73 + j41$; $Z_{13} = 37 - j26$

$Z_{33} = 82 + j67$; $Z_{23} = 63 - j9$



$Z_{in} = 23 + j15$

$G_{di} = 8$ db

Array factor shown solid

See Jour IEE Vol 93 pt 3A #3 Pg 558-614 (1946)

Problem 6.1

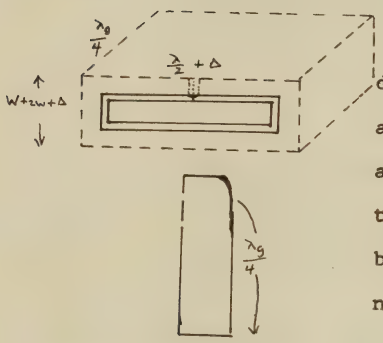
The slot antenna in Fig. 6.4 of the text should be the complement of the folded dipole constructed from strips. Thus, the radiation pattern will be that of a folded dipole of magnetic currents, which is the dual of the folded dipole of electric currents:

$$H_\theta = \left(\frac{\epsilon_0}{\mu_0}\right)^{\frac{1}{2}} E_{\theta\ \text{dipole},\lambda/2} = K\frac{e^{-j\beta r}}{r}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$Z_{slot} = \frac{1}{4}(120\pi)^2\frac{1}{Z_{dipole}} \doteq \frac{1}{4}\frac{(120\pi)^2}{4(73+j20)}$$

$$Y_{slot} \doteq \frac{16(73+j20)}{1.44\times 10^5} \doteq (8.1+j2.2)\times 10^{-3}$$

$$Z_{slot} \doteq 111 - j35 \text{ ohms}$$



A rectangular cavity with depth $\frac{1}{4}\lambda_g$ and width $\frac{1}{2}\lambda$ plus a little (Δ) should cut the admittance by half; the $\frac{1}{4}\lambda_g$ dimension (depth) could be folded over to save space, if necessary.

Problem 6.2

The first point tobe remembered is that the two sides of the horn must be designed so that they fit together and also match the .4" x .9" guide dimensions. This means that the axis projections of the slant lengths of the sides must be the same. Thus $R_{1E}(1 - \frac{b}{B}) = R_{1H}(1 - \frac{a}{A})$.

The two planes should have radiation patterns with somewhere near the same gains, so A should be greater than B, but not as great as the ratio 9/4.

A gain of 20 db means a numerical gain of 100 so we have the equation

Prob.6.2cont.

$$100 = \frac{\pi}{32}\left(G_{dE}\frac{\lambda}{A}\right)\left(G_{dH}\frac{\lambda}{B}\right)$$

Note: there is an error in the first printing of the text on page 248; the A and the B are interchanged in the formula above. Also, there is an error in the upper limit of the integral on the top of page 247, should be $\left(\frac{2}{\lambda R_1}\right)^{\frac{1}{2}}$.

Let us try to make the quantities $G_{dE}\frac{\lambda}{A}$ and $G_{dH}\frac{\lambda}{B}$ equal. Then according to the equation above they should both be about 32. A glance at the curves shows that this is not quite possible with equal slant lengths since the corresponding curves have unequal gains in the two planes.

To proceed, try a couple of slant lengths in the general range of interest . 1. Try $R_{1E}=R_{1H} = 6\lambda$. In this case $G_{dHmax}\frac{\lambda}{B} = 34$, $G_{dEmax}\frac{\lambda}{A} = 28$, product = 950, which is a little low . 2. Try $R_{1E} = R_{1H} = 8\lambda$. The individual gains are then 40 and 33 respectively and the product is 1420, which is high. In our design the product should be about 1020. We now see the approximate range of parameter values . Let us next try to settle on a specific design. Let $R_{1E} = 6\lambda$, to give $G_{dE}\frac{\lambda}{A} = 28$. Set $B = 3.7\lambda$. We should have R_{1H} somewhat greater than 6λ and the curves appear to maximize in the range 4.5 to 5λ for A, So let us try an A dimension of 4.6λ . Then $R_{1H} = R_{1E}(1 - \frac{b}{B})/(1 - \frac{a}{A}) = 6.5\lambda$. It can be shown that the directivity in the other plane is given by

$$G_{dH}\frac{\lambda}{B} = 4\pi\frac{\lambda}{A}\frac{R_1}{\lambda}\left\{\left[C(u)-C(v)\right]^2 + \left[S(u)-S(v)\right]^2\right\}$$

$$u = \frac{1}{\sqrt{2}}\left(\frac{\lambda}{A}\sqrt{\frac{R_1}{\lambda}} + \frac{A}{\lambda}\sqrt{\frac{\lambda}{R_1}}\right) \doteq 1.67\,, \quad v = \frac{1}{\sqrt{2}}\left(\frac{\lambda}{A}\sqrt{\frac{R_1}{\lambda}} - \frac{A}{\lambda}\sqrt{\frac{\lambda}{R_1}}\right) \doteq -.885$$

$C(u) = .33$, $S(u) = .57$, $C(v) = -.76$, $S(v) = -.33$; $\left[C(u)-C(v)\right]^2 \doteq 1.18$, $\left[S(u)-S(v)\right]^2 = .81$

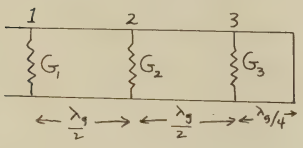
$G_{dH}\frac{\lambda}{B} = 35.5$, $G_{TOTAL} = \frac{\pi}{32} \times 28 \times 35.5 = 996$. This is probably close enough to 20db.

In Summary, the design is

$A = \frac{4.6\times 3}{2.54} = 5.42''$, $B = 4.37''$, $R_{1H} = 7.68''$, $R_{1E} = 7.09''$

Problem 6.3

Since the array is to be a broadside one, probably the easiest design to carry out is to have the shunt slots placed half a guide wavelength apart with the center slot on the opposite side of the center line from the other two. The transmission line model is then as indicated.



The admittance at position 3 is then just $Y_s = G_3$ if the slot is resonant. At point 2+, the admittance is likewise equal to G_3 since the line length is a half wavelength. At point 2- , the admittance is $Y_2 = G_2 + G_3$. Likewise, the admitance at point 1- is $Y_1 = G_1 + G_2 + G_3$. The voltage is continuous across the lumped admittances, so on the transmission line model at least, the voltage is the same on each shunt element (because of the half-wavelength spacing) (V(1+) = V(2-) = V(2+) , I(1+) = I(2-) = I(2+) - I_s).

Since the field in the slot varies as the slot admittance, to obtain the binomial amplitude distribution as slot excitation means that $G_1 = G_3 = \frac{1}{2}G_2$, if all the slots are the same distance off the center line. To get this much difference in conductance in resonant slots is difficult without using the distance to the centerline as a parameter. so instead we will make the following quantities equal: $G_1/\cos\frac{\pi x_1}{a} = G_3/\cos\frac{\pi x_3}{a} = \frac{1}{2}\,G_2/\cos\frac{\pi x_2}{a}$

Make $G_2/Y_0 \doteq .7$ with $x_2/a \doteq .28$. Then $\frac{1}{2}G_2/\cos\pi\frac{x_2}{a} = .321$

This implies that we can take $G_1/Y_0 = .28$, $x_1/a = .155$.

These parameters should put the amplitudes in the ratio 1:2:1. The terminating admittance for the guide would then be $Y/Y_0 = 1.26$, VSWR = 1.26. The slot conductances could be altered for a still better match if necessary.

Problem 6.4

The directivity of a uniformly illuminated circular aperture is $G_{di} = (\beta a)^2$, where a is the radius of the aperture.. With an aperture efficiency of 60%, we want $10 \log_{10} G_{di} \cdot .6 = 36$ or $\beta a = 81.5$ and $a = 81.5 \times 3/2\pi = 39$ cms.

The width between the nulls depends on the zeros of $J_1(\beta a \sin\theta)$, assuming uniform excitation. At the first zero $\beta a \sin\theta = 3.83$, or $\theta = 6^\circ$, so the beam width between nulls is 12°.

Problem 6.5

With the assumptions stated in the problem, the field of the full aperture and that produced by the equivalent currents in the blocking aperture have the same form; the field of the blocked aperture is then (a_1 is radius of blocking aperture)

$$E_{a\theta}\Big]_{\varphi=\frac{\pi}{2}} = j\frac{\beta}{r} E_0 e^{-j\beta r}\left[a^2 \frac{J_1(\beta a \sin\theta)}{\beta a \sin\theta} - a_1^2 \frac{J_1(\beta a_1 \sin\theta)}{\beta a_1 \sin\theta}\right]$$

If $a_1 = .1\ a$

$$E_{a\theta}\Big]_{\varphi=\frac{\pi}{2}} = j\frac{\beta E_0}{r} e^{-j\beta r} a^2\left[\frac{J_1(\beta a \sin\theta)}{\beta a \sin\theta} - (.01)\ \frac{J_1(.1\beta a \sin\theta)}{.1\beta a \sin\theta}\right]$$

Using Jahnke and Emde, or the curve in Fig. 6.12, to read off values, it is clear that the total contribution of the last term in the brackets varies from .01 to .0085 of the value of the first term, throughout the whole range of θ . The result is that the field amplitude is decreased by .01 in the range of the main lobe, the null is displaced slightly, the first side lobe is decreased slightly and the second is increased slightly. The effect would be barely noticeable in a sketch of the pattern. To compute the gain of the blocked aperture, note $|E_{axis}| = \frac{\beta a^2 E_0}{2r}[1-.01] = .99\frac{\beta a^2 E_0}{2r}$ while the power coming through the aperture is $P_{av} = \frac{1}{2} E_0^2 \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} \pi [a^2 - .01 a^2] = \frac{1}{2} E_0 \left(\frac{\epsilon}{\mu}\right)^{\frac{1}{2}} \pi a^2 [.99]$. It follows from its definition that $G_{di} \doteq .99(\beta a)^2$. In general $G_{di} \doteq (\beta a)^2\left[1 - \left(\frac{a_1}{a}\right)^2\right]$.

Problem 7.1

A log-periodic dipole antenna is to be designed to operate from 50 to 250 MHz. In that frequency range, we might use 5/8" OD tubing at the lower frequencies and 3/8" OD tubing at the higher frequencies. At 50 MHz., a half wavelength is 3 meters so the length to diameter ratio of the elements will be in the vicinity of 189 . This is a reasonable value.

Let us try to design for maximum gain. Looking at the curves of Fig.7.17, it appears that 12 db should be possible if we take a relative spacing of .18 and a scale factor, τ , of .965 . The angle, α , is then

$$\alpha = \tan^{-1}\frac{1 - .965}{4} = 2.75^{o}$$

From the nomograph Fig. 7.18, $B_{ar} = 1.295$, so we will design for a bandwidth $B_s = 5 \times 1.295 = 6.47$, that is we will design for operation from 44 to 282 MHz. The longest element is then a half wavelength at 44 MHz or 3.4 meters . The length of the structure is then

$L = 1.7(1 - \frac{1}{6.47})\cot$ =29.5 meters. This is a rather ridiculously long structure for the intended application. But this is the problem with designs whose prime objective is to produce high gains. The number of elements is $N = 1 + \log 6.47/\log(1/.965) = 53$. This would be a TV antenna to make the neighbors sit up and take notice.

Actually for VHF TV, performance is required only from 54 to 108 and 174 to 216, so as a first thought we might try to design two LParrays for the two bands and try to put them together. This would cut both size and number of elements. But notice also that the high band is a factor of three higher than the low band; this means that a (compromize) design for the low band might be acceptable for the high band. The active region would then be in the vicinity of the elements whose lengths were $(3/2)\lambda$ at the frequency of operation.

Problem 8.1

If the aperture antenna is excited as a transmitter, it produces fields E_r and H_r. Let the primary source for these fields be a voltage source at a set of terminals interior to the surface of the aperture antenna. Next imagine that a magnetic current source, equivalent to the voltage source, replaces the voltage source. Then from equation 3

$$I_{sc} = \frac{1}{V_g} \int J_a \cdot E_r \, dv$$

where I_{sc} is the short circuit current at the interior terminals, E_r is the field produced by the magnetic current source at the location of J_a, the latter being the source of the incident field. However, E_r is also produced by the Huygens sources on the closed surface surrounding the primary source and including the aperture. Thus,

$$\int J_a \cdot E_r \, dv = \int (E_a \cdot J_h - H_a \cdot M_h) \, dv$$

But the Huygens sources J_h, M_h act in free space so $E_a = E_{inc}$, $H_a = H_{inc}$ and

$$V_g I_{sc} = \int (E_{inc} \cdot J_h - H_{inc} \cdot M_h) \, dv$$

which was to be demonstrated.

Problem 8.2

Imagine a point current source, I_g, at the input terminals of the receiving antenna. By reciprocity,

$$V_{oc} I_g = \int E_r \cdot \mathcal{J}_a \, dv$$

where E_r is the field produced by I_g at the position of $\mathcal{J}_a$, the source of the incident field. (ie., $\int E_a \cdot \mathcal{J}_g dv = V_{oc} I_g$) But E_r could also be produced by the combination of I_g and the induced currents J_i acting in free space (the currents J_i are the currents produced on the structure when I_g is activated.

Prob.8.2cont.

Call $\mathcal{J}_g + J_i$ by the name J_t, the transmitting current distribution. Then $\int E_r \cdot \mathcal{J}_a \, dv = \int E_{inc} \cdot J_t \, dv$ or

$$V_{oc} = \frac{1}{I_{in}} \int E_{inc} \cdot J_t \, dv$$

where $I_{in} = I_g$ = the current at the input terminals.

Problem 8.3

First consider the power received by the parabolic reflector. Since it has a gain of 40 db, the effective aperture is $A_{eff} = 10^4 \frac{(.3)^2}{4\pi}$ at 1 GHz. The received power is then

$$P_{signal} = 10^4 \frac{(.3)^2}{4\pi} \frac{1}{4\pi (200 \times 10^3)^2} \times 1.5$$

The noise power delivered is $P_{noise} = k \, T_a \, df$ where T_a is the antenna temperature:

$$T_a = \frac{1}{4\pi} \iint G(\theta,\varphi) \, T_b(\theta,\varphi) \, \sin\theta \, d\theta \, d\varphi$$

We will suppose that the parabolic reflector has a circular aperture and in the absence of other information assume an aperture efficiency of unity. Thus, since the gain is equal to $(\beta a)^2$, $(\beta a) = 10^2$. If the sun is 5^o off the axis, the argument $(\beta a \sin\theta) = 8.72$. From figure 6.13, this puts the sun in the second side lobe at which point the gain relative to the directivity is .0038 or the gain is 38. We might suppose that the rest of the environment (a combination of galactic noise, atmospheric noise, and noise from the earth is at an average temperature of 150^o. Then $T_a = \frac{1}{4\pi} \iint G(\theta,\varphi) \, 150^\circ \sin\theta \, d\theta \, d\varphi + \frac{1}{4\pi} (38) 3 \times 10^{5^\circ} (1.75) \times 10^{-4} \doteq 150^o + 280^o \doteq 430^o$

or $\frac{\text{Signal}}{\text{Noise}} = \frac{13.5 \times 10^2}{64\pi^2 \times 10^{10}} \frac{1}{1.38 \times 10^{-23} \times 430 \times 1 \times 10^6} \simeq 3.6 \times 10^4$

or about 45 db.

Problem 8.4

At 100 kHz , the wavelength is 3000 meters, so the 800 foot monopole is a relatively small antenna. The directivity over perfect ground would be 3. Taking into account the stated efficiency of 70% , the power density in the distant field ($\theta = 90^{o}$) with zero ground loss would be

$$PD = \frac{.7\ P_{in}\ \ 3}{4\ \ (600x1.609x10^{3})^{2}}$$

Since the sky wave field strength varies, let us be conservative and base the calculation on ground wave propagation. According to ground wave curves such as appear in the ITT Reference Data for Radio Engineers, 4^{th} ed., pg 714, the ground wave field strengths could be expected to be about 20 db below the free space inverse square fields. So accounting for ground wave loss

$$PD = P_{in} 2.1 / 4\pi x 36 x 2.58 x 10^{12}$$

We will assume that the field will be picked up by a small whip antenna with no losses. In this case, the power delivered will be $P_{sig} = PD\ A_{eff} = PDx\ 3\ \lambda^{2}/4\pi$

or $P_{sig} = P_{in} x\ 6.3x\ 9x\ 10^{-6} / 16\pi^{2} x\ 36x\ 2.58$

The noise power delivered will be $P_{noise} = k\ T_{a}\ df$

From Fig. 8.5, the equivalent atmospheric noise temperature could be expected to be as high as 120 db above 288^{oK} or $T_{a} = 2.88 \times 10^{14}$. The required signal to noise ratio is 20 db so we have

$$100 = \frac{P_{in} \times 6.3 \times 10^{-6}}{64\pi^{2} \times 2.58} \quad \frac{1}{1.38 \times 10^{-23} \times 2.88 \times 10^{14} \times 3 \times 10^{3}}$$

From which

$$P_{in} \approx 300\ kW$$

NOTES

NOTES

NOTES

68971